A
SCHOOL GEOMETRY

PARTS I AND II

Part I Lines and Angles Rectilineal Figures
Part II Areas of Rectilineal Figures

(Containing the substance of Euclid, Book I)

BY

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THIRD EDITION REVISED

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PREFACE

The present work provides a course of Elementary Geometry based on the recommendations of the Mathematical Association and on the schedule recently proposed and adopted at Cambridge

The principles which governed these proposals have been confirmed by the issue of revised schedules for all the more important Examinations, and they are now so generally accepted by teachers that they need no discussion here It is enough to note the following points

(i) We agree that a pupil should gain his first geometrical ideas from a short preliminary course of a practical and experimental character A suitable introduction to the present book would consist of Easy Exercises in Drawing to illustrate the subject matter of the Definitions, Measurements of Lines and Angles, Use of Compasses and Protractor, Problems on Bisection, Perpendiculars, and Parallels, Use of Set Squares, The Construction of Triangles and Quadrilaterals These problems should be accompanied by informal explanation, and the results verified by measurement Concurrently, there should be a series of exercises in Drawing and Measurement designed to lead inductively to the more important Theorems of Part I [Euc. I. 1-34].* While strongly advocating some such introductory lessons, we may point out that our book, as far as it goes, is complete in itself, and from the first is illustrated by numerical and graphical examples of the easiest types. Thus, throughout the whole work, a graphical and experimental course is provided side by side with the usual deductive exercises

(ii) Theorems and Problems are arranged in separate but parallel courses, intended to be studied *pari passu* This arrangement is made possible by the use, now generally sanctioned, of *Hypothetical Constructions* These, before being employed in the text, are carefully specified, and referred to the Axioms on which they depend.

* Such an introductory course is now furnished by our *Lessons in Experimental and Practical Geometry*

(iii) The subject is placed on the basis of *Commensurable Magnitudes*. By this means, certain difficulties which are wholly beyond the grasp of a young learner are postponed, and a wide field of graphical and numerical illustration is opened. Moreover the fundamental Theorems on Areas (hardly less than those on Proportion) may thus be reduced in number, greatly simplified, and brought into line with practical applications.

(iv) An attempt has been made to curtail the excessive body of text which the demands of Examinations have hitherto forced as "bookwork" on a beginner's memory. Even of the Theorems here given a certain number (which we have distinguished with an asterisk) might be omitted or postponed at the discretion of the teacher. And the formal propositions for which—as such—teacher and pupil are held responsible, might perhaps be still further limited to those which make the landmarks of Elementary Geometry. Time so gained should be used in getting the pupil to *apply* his knowledge; and the working of examples should be made as important a part of a lesson in Geometry as it is so considered in Arithmetic and Algebra.

Though we have not always followed Euclid's order of Propositions, we think it desirable for the present, in regard to the subject matter of Euclid Book I. to preserve the essentials of his logical sequence. Our departure from Euclid's treatment of Areas has already been mentioned; the only other important divergence in this section of the work is the position of I. 26 (Theorem 17), which we place after I. 32 (Theorem 16), thus getting rid of the tedious and uninstructive *Second Case*. In subsequent Parts a freer treatment in respect of logical order has been followed.

As regards the presentment of the propositions, we have constantly kept in mind the needs of that large class of students, who, without special aptitude for mathematical study, and under no necessity for acquiring technical knowledge, may and do derive real intellectual advantage from lessons in pure deductive reasoning. Nothing has as yet been devised as effective for this purpose as the Euclidean form of proof; and in our opinion no excuse is needed for treating the earlier proofs with that fulness which we have always found necessary in our experience as teachers.

The examples are numerous and for the most part easy. They have been very carefully arranged, and are distributed throughout the text in immediate connection with the propositions on which they depend. A special feature is the large number of examples involving graphical or numerical work. The answers to these have been printed on perforated pages, so that they may easily be removed if it is found that access to numerical results is a source of temptation in examples involving measurement.

We are indebted to several friends for advice and suggestions. In particular we wish to express our thanks to Mr H. C. Playne and Mr H. C. Beaven of Clifton College for the valuable assistance they have rendered in reading the proof sheets and checking the answers to some of the numerical exercises.

H. S. HALL.
F. H. STEVENS.

November, 1903.

PREFATORY NOTE TO THE THIRD EDITION

In the present edition some further steps have been taken towards the curtailment of bookwork by reducing certain less important propositions (e.g. Euclid I. 22, 43, 44) to the rank of exercises. Room has thus been found for more numerical and graphical exercises, and experimental work such as that leading to the Theorem of Pythagoras.

Theorem 22 (page 62), in the shape recommended in the Cambridge Schedule, replaces the equivalent proposition given as *Additional Theorem A* (page 60) in previous editions.

In the case of a few problems (e.g. Problems 23, 28, 29) it has been thought more instructive to justify the construction by a preliminary *analysis* than by the usual formal proof.

H. S. HALL.
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March, 1904.

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PART I

PART II

GEOMETRY

PART I.

Axioms

All mathematical reasoning is founded on certain simple principles, the truth of which is so evident that they are accepted without proof These self-evident truths are called **Axioms.**

For instance

Things which are equal to the same thing are equal to one another

The following axioms, corresponding to the first four Rules of Arithmetic, are among those most commonly used in geometrical reasoning

Addition. *If equals are added to equals, the sums are equal*

Subtraction. *If equals are taken from equals, the remainders are equal*

Multiplication. *Things which are the same multiples of equals are equal to one another*

For instance *Doubles of equals are equal to one another*

Division. *Things which are the same parts of equals are equal to one another*

For instance *Halves of equals are equal to one another*

The above Axioms are given as instances, and not as a complete list, of those which will be used. They are said to be *general*, because they apply equally to magnitudes of all kinds. Certain special axioms relating to geometrical magnitudes only will be stated from time to time as they are required.

DEFINITIONS AND FIRST PRINCIPLES

Every beginner knows in a general way what is meant by a point, a line, and a surface But in geometry these terms are used in a strict sense which needs some explanation

1 A point has position, but is said to have *no magnitude*

This means that we are to attach to a point no idea of size either as to *length* or *breadth*, but to think only where it is situated A dot made with a sharp pencil may be taken as roughly representing a point, but small as such a dot may be, it still has *some* length and breadth, and is therefore not actually a geometrical point. The smaller the dot however, the more nearly it represents a point.

2 A line has length, but is said to have *no breadth*

A line is traced out by a moving point If the point of a pencil is moved over a sheet of paper, the trace left represents a line. But such a trace, however finely drawn, has some degree of breadth, and is therefore not itself a true geometrical line The finer the trace left by the moving pencil point, the more nearly will it represent a line

3 Proceeding in a similar manner from the idea of a line to the idea of a surface, we say that

A surface has length and breadth, but *no thickness*

And finally,

A solid has length, breadth, and thickness

Solids, surfaces, lines and points are thus related to one another

(i) A solid is bounded by surfaces

(ii) A surface is bounded by lines, and surfaces meet in lines

(iii) A line is bounded (or terminated) by points, and lines meet in points.

4 A line may be straight or curved

A straight line has the same direction from point to point throughout its whole length

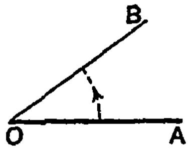
A curved line changes its direction continually from point to point

AXIOM *There can be only one straight line joining two given points* that is,

Two straight lines cannot enclose a space

5 A plane is a *flat* surface, the test of flatness being that if any two points are taken in the surface, the straight line between them lies wholly in that surface

6 When two straight lines meet at a point, they are said to form an angle



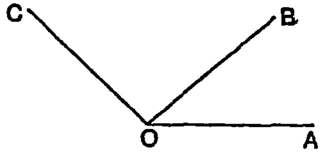
The straight lines are called the arms of the angle, the point at which they meet is its vertex.

The magnitude of the angle may be thus explained

Suppose that the arm OA is fixed, and that OB *turns about the point* O (as shewn by the arrow) Suppose also that OB began its turning from the position OA Then the size of the angle AOB is measured by the *amount of turning* required to bring the revolving arm from its first position OA into its subsequent position OB

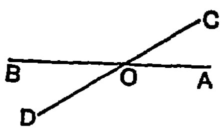
Observe that the size of an angle does not in any way depend on the length of its arms

Angles which lie on either side of a common arm are said to be adjacent

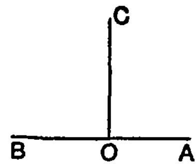


For example, the angles AOB, BOC, which have the common arm OB, are adjacent.

When two straight lines such as AB, CD cross one another at O, the angles COA, BOD are said to be vertically opposite The angles AOD, COB are also vertically opposite to one another



7 When one straight line stands on another so as to make the adjacent angles equal to one another, each of the angles is called a **right angle**, and each line is said to be **perpendicular** to the other



AXIOMS (i) *If* O *is a point in a straight line* AB, *then a line* OC, *which turns about* O *from the position* OA *to the position* OB, *must pass through one position, and only one, in which it is perpendicular to* AB

(ii) *All right angles are equal*

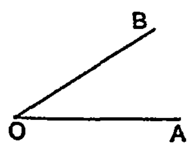
A right angle is divided into 90 equal parts called **degrees** (°), each degree into 60 equal parts called **minutes** (′), each minute into 60 equal parts called **seconds** (″)

In the above figure, if OC revolves about O from the position OA into the position OB, it turns through *two right angles*, or 180°

If OC makes a *complete revolution* about O, starting from OA and returning to its original position, it turns through *four right angles*, or 360°

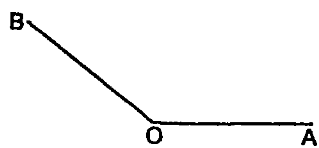
8 An angle which is less than one right angle is said to be **acute**

That is, an acute angle is less than 90°



9 An angle which is greater than *one* right angle, but less than *two* right angles, is said to be **obtuse**

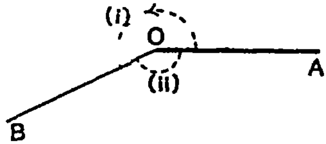
That is, an obtuse angle lies between 90° and 180°



10 If one arm OB of an angle turns until it makes *a straight line* with the other arm OA, the angle so formed is called a **straight angle**

A straight angle = 2 right angles = 180°

11 An angle which is greater than *two* right angles, but less than *four* right angles, is said to be **reflex.**

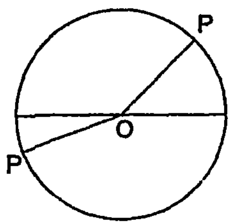


That is, a reflex angle lies between 180° and 360°

NOTE. When two straight lines meet, *two* angles are formed, one greater, and one less than two right angles The first arises by supposing OB to have revolved from the position OA the longer way round, marked (i), the other by supposing OB to have revolved the shorter way round, marked (ii) Unless the contrary is stated, the angle between two straight lines will be considered to be that which is less than two right angles

12 Any portion of a plane surface bounded by one or more lines is called a **plane figure**

13 A **circle** is a plane figure contained by a line traced out by a point which moves so that its distance from a certain fixed point is always the same



Here the point P moves so that its distance from the fixed point O is always the same

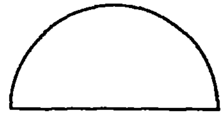
The fixed point is called the **centre**, and the bounding line is called the **circumference**

14 A **radius** of a circle is a straight line drawn from the centre to the circumference It follows that all radii of a circle are equal

15 A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference

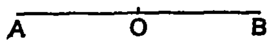
16 An **arc** of a circle is any part of the circumference

17 A semi-circle is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter



18 To bisect means to divide into two *equal* parts

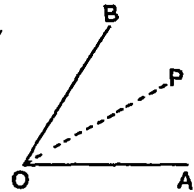
Axioms (i) *If a point* O *moves from* A *to* B *along the straight line* AB, *it must pass through one position in which it divides* AB *into two equal parts*



That is to say

Every finite straight line has a point of bisection

(ii) *If a line* OP, *revolving about* O *turns from* OA *to* OB, *it must pass through one position in which it divides the angle* AOB *into two equal parts*



That is to say

Every angle may be supposed to have a line of bisection.

Hypothetical Constructions

From the Axioms attached to Definitions 7 and 18, it follows that we may suppose

(i) *A straight line to be drawn perpendicular to a given straight line from any point in it*

(ii) *A finite straight line to be bisected at a point*

(iii) *An angle to be bisected by a line*

Superposition and Equality

Axiom *Magnitudes which can be made to coincide with one another are equal*

This axiom implies that any line, angle, or figure, may be taken up from its position, and without change in size or form, laid down upon a second line, angle, or figure, for the purpose of comparison, and it states that two such magnitudes are equal when one can be exactly placed over the other without overlapping

This process is called **superposition**, and the first magnitude is said to be applied to the other

Postulates

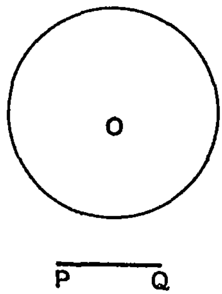
In order to draw geometrical figures certain instruments are required These are, for the purposes of this book, (i) *a straight ruler* (ii) *a pair of compasses* The following Postulates (or requests) claim the use of these instruments, and assume that with their help the processes mentioned below may be duly performed

Let it be granted

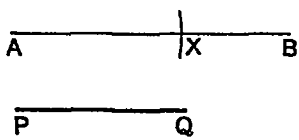
1 *That a straight line may be drawn from any one point to any other point*

2 *That a* FINITE *(or terminated) straight line may be* PRODUCED *(that is, prolonged) to any length in that straight line*

3 *That a circle may be drawn with any point as centre and with a radius of any length*



NOTES (i) Postulate 3, as stated above, implies that we may adjust the compasses to the length of any straight line PQ, and with a radius of this length draw a circle with any point O as centre That is to say, the compasses may be used to *transfer distances* from one part of a diagram to another



(ii) *Hence from* AB, *the greater of two straight lines, we may cut off a part equal to* PQ *the less*

For if with centre A, and radius equal to PQ, we draw an arc of a circle cutting AB at X, it is obvious that AX is equal to PQ.

INTRODUCTORY

1 Plane geometry deals with the properties of such lines and figures as may be drawn on a plane surface

2 The subject is divided into a number of separate discussions, called **propositions**

Propositions are of two kinds, **Theorems** and **Problems**

A **Theorem** proposes to prove the truth of some geometrical statement

A **Problem** proposes to perform some geometrical construction, such as to draw some particular line, or to construct some required figure

3 A Proposition consists of the following parts

The *General Enunciation*, the *Particular Enunciation*, the *Construction*, and the *Proof*

(i) The **General Enunciation** is a preliminary statement, describing in general terms the purpose of the proposition

(ii) The **Particular Enunciation** repeats in special terms the statement already made, and refers it to a diagram, which enables the reader to follow the reasoning more easily

(iii) The **Construction** then directs the drawing of such straight lines and circles as may be required to effect the purpose of a problem, or to prove the truth of a theorem

(iv) The **Proof** shews that the object proposed in a problem has been accomplished, or that the property stated in a theorem is true

4 The letters Q E D are appended to a theorem, and stand for **Quod erat Demonstrandum**, *which was to be proved*

5 A **Corollary** is a statement the truth of which follows readily from an established proposition, it is therefore appended to the proposition as an inference or deduction, which usually requires no further proof

6 The following symbols and abbreviations are used in the text of this book

In Part I

	for	therefore,	$\angle$	*for*	angle,
=	„	is, or are, equal to,	$\triangle$	„	triangle

After Part I

pt	*for*	point,	perp	*for*	perpendicular,
st line	„	straight line,	par^{m}	„	parallelogram,
rt $\angle$	„	right angle,	rectil	„	rectilineal,
par^{l} (or $\parallel$)	„	parallel,	$\odot$	„	circle,
sq	„	square,	$\bigcirc^{ce}$	„	circumference,

and all obvious contractions of commonly occurring words, such as opp, adj, diag, etc, *for* opposite, adjacent, diagonal, etc

[For convenience of oral work, and to prevent the rather common abuse of contractions by beginners, the above code of signs has been introduced gradually, and at first somewhat sparingly]

In numerical examples the following abbreviations will be used

m	*for*	metre,	cm	*for*	centimetre
mm	„	millimetre	km	„	kilometre

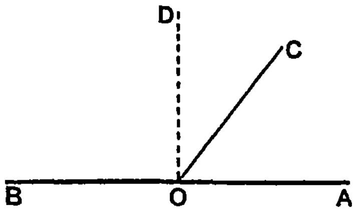
Also inches are denoted by the symbol (")

Thus 5" means 5 inches

ON LINES AND ANGLES

THEOREM 1 [Euclid I 13]

The adjacent angles which one straight line makes with another straight line on one side of it, are together equal to two right angles



Let the straight line CO make with the straight line AB the adjacent $\angle^s$ AOC, COB

It is required to prove that the $\angle^s$ AOC, COB *are together equal to two right angles*

Suppose OD is at right angles to BA.

Proof Then the $\angle^s$ AOC, COB together
= the three $\angle^s$ AOC, COD, DOB

Also the $\angle^s$ AOD, DOB together
= the three $\angle^s$ AOC, COD, DOB

the $\angle^s$ AOC, COB together = the $\angle^s$ AOD, DOB
= two right angles

Q E D

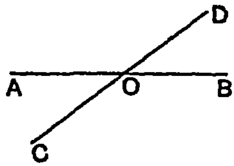
PROOF BY ROTATION

Suppose a straight line revolving about O turns from the position OA into the position OC, and thence into the position OB, that is, let the revolving line turn in succession through the $\angle^s$ AOC, COB

Now in passing from its first position OA to its final position OB, the revolving line turns through two right angles, for AOB is a straight line

Hence the $\angle^s$ AOC, COB together = two right angles

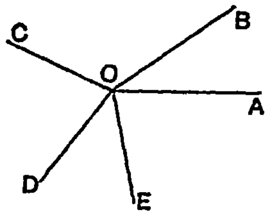
Corollary 1 *If two straight lines cut one another, the four angles so formed are together equal to four right angles*



For example,

$$\angle BOD + \angle DOA + \angle AOC + \angle COB = 4 \text{ right angles}$$

Corollary 2 *When any number of straight lines meet at a point, the sum of the consecutive angles so formed is equal to four right angles*



For a straight line revolving about O, and turning in succession through the $\angle^s$ AOB, BOC, COD, DOE, EOA, will have made one complete revolution, and therefore turned through four right angles

DEFINITIONS

(i) Two angles whose sum is *two* right angles, are said to be **supplementary** and each is called the **supplement** of the other

Thus in the Fig of Theor 1 the angles AOC, COB are supplementary Again the angle 123° is the supplement of the angle 57°

(ii) Two angles whose sum is *one* right angle are said to be **complementary**, and each is called the **complement** of the other

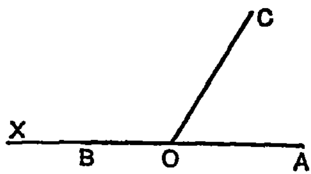
Thus in the Fig of Theor 1 the angle DOC is the complement of the angle AOC Again angles of 34° and 56° are complementary

Corollary 3 (i) *Supplements of the same angle are equal*

(ii) *Complements of the same angle are equal*

THEOREM 2 [Euclid I 14]

If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, then these two straight lines are in one and the same straight line



At O in the straight line CO let the two straight lines OA, OB, on opposite sides of CO, make the adjacent $\angle^s$ AOC, COB together equal to two right angles (that is, let the adjacent $\angle^s$ AOC, COB be supplementary)

It is required to prove that OB *and* OA *are in the same straight line*

Produce AO beyond O to any point X it will be shewn that OX and OB are the same line

Proof Since by construction AOX is a straight line,
the $\angle$ COX is the supplement of the $\angle$ COA *Theor* 1.

But, by hypothesis,
the $\angle$ COB is the supplement of the $\angle$ COA

the $\angle$ COX = the $\angle$ COB,
OX and OB are the same line

But, by construction, OX is in the same straight line with OA,
hence OB is also in the same straight line with OA

Q E D

EXERCISES

1 Write down the supplements of *one half* of a right angle, *four thirds* of a right angle; also of 16°, 149°, 83°, 101° 15′.

2 Write down the complement of *two fifths* of a right angle; also of 27°, 38° 16′, and 41° 20′ 30″.

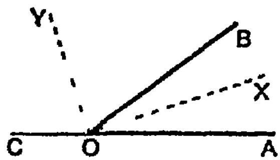
3 If two straight lines intersect forming four angles of which one is known to be a right angle, prove that the other three are also right angles.

4 In the triangle ABC the angles ABC, ACB are given equal. If the side BC is produced both ways, shew that the exterior angles so formed are equal.

5 In the triangle ABC the angles ABC, ACB are given equal. If AB and AC are produced beyond the base, shew that the exterior angles so formed are equal.

DEFINITION. The lines which bisect an angle and the adjacent angle made by producing one of its arms are called the internal and external bisectors of the given angle.

Thus in the diagram, OX and OY are the internal and external bisectors of the angle AOB.



6 Prove that the bisectors of the adjacent angles which one straight line makes with another contain a right angle. That is to say, *the internal and external bisectors of an angle are at right angles to one another.*

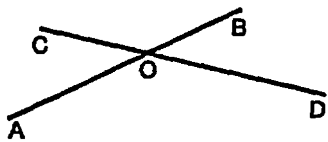
7 Shew that the angles AOX and COY in the above diagram are complementary.

8 Shew that the angles BOX and COX are supplementary; and also that the angles AOY and BOY are supplementary.

9 If the angle AOB is 35°, find the angle COY.

Theorem 3 [Euclid I 15]

If two straight lines cut one another, the vertically opposite angles are equal



Let the straight lines AB, CD cut one another at the point O

It is required to prove that

(i) *the* ∠ AOC = *the* ∠ DOB,
(ii) *the* ∠ COB = *the* ∠ AOD

Proof Because AO meets the straight line CD,
the adjacent ∠s AOC, AOD together = two right angles,
that is, the ∠ AOC is the supplement of the ∠ AOD

Again, because DO meets the straight line AB,
the adjacent ∠s DOB, AOD together = two right angles,
that is, the ∠ DOB is the supplement of the ∠ AOD

Thus each of the ∠s AOC, DOB is the supplement of the ∠ AOD,
the ∠ AOC = the ∠ DOB

Similarly, the ∠ COB = the ∠ AOD

Q E D

PROOF BY ROTATION

Suppose the line COD to revolve about O until OC turns into the position OA Then at the same moment OD must reach the position OB (for AOB and COD are *straight*)

Thus the same amount of turning is required to close the ∠ AOC as to close the ∠ DOB

the ∠ AOC = the ∠ DOB

EXERCISES ON ANGLES

(Numerical.)

1 Through what angles does the minute hand of a clock turn in (i) 5 minutes, (ii) 21 minutes, (iii) $43\frac{1}{3}$ minutes, (iv) 14 min 10 sec? And how long will it take to turn through (v) 66°, (vi) 222°?

2 A clock is started at noon through what angles will the hour hand have turned by (i) 3 45, (ii) 10 minutes past 5? And what will be the time when it has turned through $172\frac{1}{2}$°?

3 The earth makes a complete revolution about its axis in 24 hours Through what angle will it turn in 3 hrs 20 min, and how long will it take to turn through 130°?

4 In the diagram of Theorem 3

(i) If the $\angle$AOC=35°, write down (without measurement) the value of each of the $\angle$s COB, BOD, DOA

(ii) If the $\angle$s COB, AOD together make up 250°, find each of the $\angle$s COA, BOD

(iii) If the $\angle$s AOC, COB, BOD together make up 274°, find each of the four angles at O

(Theoretical.)

5 If from O a point in AB two straight lines OC, OD are drawn on opposite sides of AB so as to make the angle COB equal to the angle AOD, shew that OC and OD are in the same straight line

6 Two straight lines AB, CD cross at O If OX is the bisector of the angle BOD, prove that XO produced bisects the angle AOC

7 Two straight lines AB, CD cross at O If the angle BOD is bisected by OX, and AOC by OY, prove that OX, OY are in the same straight line.

8 If OX bisects an angle AOB, shew that, by folding the diagram about the bisector, OA may be made to coincide with OB

How would OA fall with regard to OB, if

(i) the $\angle$AOX were greater than the $\angle$XOB,
(ii) the $\angle$AOX were less than the $\angle$XOB?

9 AB and CD are straight lines intersecting at right angles at O, shew by folding the figure about AB, that OC may be made to fall along OD

10 A straight line AOB is drawn on paper, which is then folded about O, so as to make OA fall along OB, shew that the crease left in the paper is perpendicular to AB

ON TRIANGLES

1 Any portion of a plane surface bounded by one or more lines is called a **plane figure**

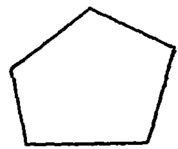
The sum of the bounding lines is called the **perimeter** of the figure The amount of surface enclosed by the perimeter is called the **area.**

2 **Rectilineal figures** are those which are bounded by straight lines

3 A **triangle** is a plane figure bounded by *three* straight lines

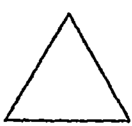
4 A **quadrilateral** is a plane figure bounded by *four* straight lines

5 A **polygon** is a plane figure bounded by more than four straight lines



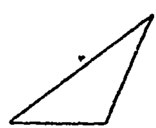
6 A rectilineal figure is said to be
equilateral, when all its sides are equal ,
equiangular, when all its angles are equal ,
regular, when it is both equilateral and equiangular

7 Triangles are thus classified with regard to their sides
A triangle is said to be
equilateral, when *all* its sides are equal ,
isosceles, when *two* of its sides are equal ,
scalene, when its sides are all unequal



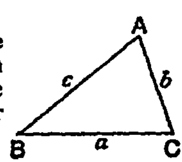
Equilateral Triangle.

Isosceles Triangle.



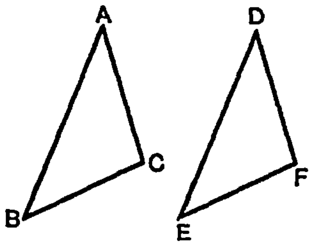
Scalene Triangle.

In a triangle ABC, the letters A, B, C often denote the *magnitude* of the several angles (as measured in degrees) , and the letters a, b, c the *lengths* of the opposite sides (as measured in inches, centimetres, or some other unit of length)



THEOREM 4 [Euclid I 4]

If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles included by those sides equal, then the triangles are equal in all respects



Let ABC, DEF be two triangles in which

AB = DE,
AC = DF,

and the included angle BAC = the included angle EDF

It is required to prove that the △ABC = *the* △DEF *in all respects*

Proof Apply the △ABC to the △DEF,
so that the point A falls on the point D,
and the side AB along the side DE

Then because AB = DE,
the point B must coincide with the point E

And because AB falls along DE,
and the ∠BAC = the ∠EDF,
AC must fall along DF

And because AC = DF,
the point C must coincide with the point F

Then since B coincides with E, and C with F,
the side BC must coincide with the side EF

Hence the △ABC coincides with the △DEF,
and is therefore equal to it in all respects

Q E

Obs In this Theorem we must carefully observe what is *given* and what is *proved*

Given that $\begin{cases} AB = DE, \\ AC = DF, \\ \text{and the } \angle BAC = \text{the } \angle EDF \end{cases}$

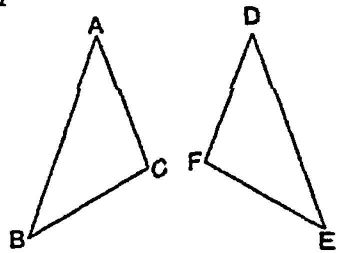
From these data we prove that the triangles coincide on superposition

Hence we *conclude* that $\begin{cases} BC = EF, \\ \text{the } \angle ABC = \text{the } \angle DEF, \\ \text{and the } \angle ACB = \text{the } \angle DFE, \end{cases}$

also that the triangles are equal in area

Notice that the angles which are proved equal in the two triangles are opposite to sides which were given equal

NOTE. The adjoining diagram shews that in order to make two congruent triangles coincide, it may be necessary to *reverse*, that is, turn over one of them before superposition



EXERCISES

1 *Shew that the bisector of the vertical angle of an isosceles triangle* (i) *bisects the base* (ii) *is perpendicular to the base.*

2 *Let* O *be the middle point of a straight line* AB, *and let* OC *be perpendicular to it* *Then if* P *is any point in* OC, *prove that* PA = PB

3 Assuming that the four sides of a square are equal, and that its angles are all right angles, prove that in the square ABCD, the diagonals AC, BD are equal

4 ABCD is a square, and L, M, and N are the middle points of AB, BC, and CD prove that

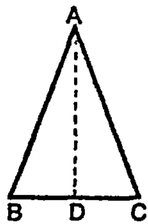
(i) LM = MN (ii) AM = DM
(iii) AN = AM (iv) BN = DM

[Draw a separate figure in each case]

5 ABC is an isosceles triangle from the equal sides AB, AC two equal parts AX, AY are cut off, and BY and CX are joined. Prove that BY = CX

THEOREM 5 [Euclid I 5]

The angles at the base of an isosceles triangle are equal



Let ABC be an isosceles triangle, in which the side AB = the side AC

It is required to prove that the ∠ ABC = *the* ∠ ACB

Suppose that AD is the line which bisects the ∠ BAC, and let it meet BC in D

1st Proof Then in the △ˢ BAD, CAD,

because
- BA = CA,
- AD is common to both triangles,
- and the included ∠ BAD = the included ∠ CAD,

the triangles are equal in all respects, *Theor* 4
so that the ∠ ABD = the ∠ ACD

Q E D

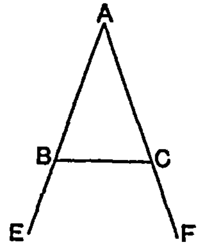
2nd Proof Suppose the △ ABC to be folded about AD

Then since the ∠ BAD = the ∠ CAD,
AB must fall along AC

And since AB = AC,
B must fall on C, and consequently DB on DC

the ∠ ABD will coincide with the ∠ ACD, and is therefore equal to it

Q E D



COROLLARY 1 *If the equal sides* AB, AC *of an isosceles triangle are produced, the exterior angles* EBC, FCB *are equal, for they are the supplements of the equal angles at the base*

COROLLARY 2 *If a triangle is equilateral, it is also equiangular*

DEFINITION A figure is said to be symmetrical about a line when, on being folded about that line, the parts of the figure on each side of it can be brought into coincidence

The straight line is called an axis of symmetry.

That this may be possible, it is clear that the two parts of the figure must have the same size and shape, and must be similarly placed with regard to the axis

Theorem 5 proves that *an isosceles triangle is symmetrical about the bisector of its* VERTICAL *angle*

An equilateral triangle is symmetrical about the bisector of ANY ONE *of its angles*

EXERCISES

1 ABCD is a four sided figure whose sides are all equal, and the diagonal BD is drawn shew that

(i) the angle ABD = the angle ADB,
(ii) the angle CBD = the angle CDB,
(iii) the angle ABC = the angle ADC

2. ABC, DBC are two isosceles triangles drawn on the same base BC, but on *opposite* sides of it prove (by means of Theorem 5) that

the angle ABD = the angle ACD

3 ABC, DBC are two isosceles triangles drawn on the same base BC and on *the same* side of it employ Theorem 5 to prove that

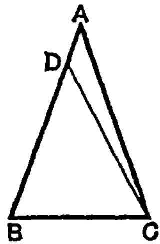
the angle ABD = the angle ACD

4 AB, AC are the equal sides of an isosceles triangle ABC, and L, M, N are the middle points of AB, BC, and CA respectively prove that

(i) LM = NM (ii) BN = CL.
(iii) the angle ALM = the angle ANM

THEOREM 6 [Euclid I 6]

If two angles of a triangle are equal to one another, then the sides which are opposite to the equal angles are equal to one another



Let ABC be a triangle in which

the $\angle$ ABC = the $\angle$ ACB

It is required to prove that the side AC = *the side* AB

If AC and AB are not equal, suppose that AB is the greater

From BA cut off BD equal to AC

Join DC

Proof Then in the $\triangle^s$ DBC, ACB,

because $\begin{cases} \text{DB} = \text{AC}, \\ \text{BC is common to both,} \\ \text{and the included } \angle \text{ DBC} = \text{the included } \angle \text{ ACB} \end{cases}$

the $\triangle$ DBC = the $\triangle$ ACB in area, *Theor* 4

the part equal to the whole, which is absurd

AB is not unequal to AC,

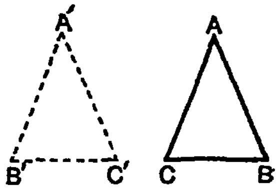
that is, AB = AC

Q E.D

COROLLARY *Hence if a triangle is equiangular it is also equilateral*

NOTE ON THEOREMS 5 AND 6

Theorems 5 and 6 may be verified experimentally by cutting out the given △ABC, and, after turning it over, fitting it *thus reversed* into the vacant space left in the paper



Suppose A'B'C' to be the original position of the △ABC, and let ACB represent the triangle when reversed

In Theorem 5, it will be found on applying A to A' that C may be made to fall on B', and B on C

In Theorem 6, on applying C to B' and B to C' we find that A will fall on A'

In either case the given triangle *reversed* will coincide with its own "trace," so that the side and angle on the *left* are respectively equal to the side and angle on the *right*

NOTE ON A THEOREM AND ITS CONVERSE

The enunciation of a theorem consists of two clauses The first clause tells us what we are to *assume*, and is called the **hypothesis**, the second tells us what *it is required to prove*, and is called the **conclusion.**

For example, the enunciation of Theorem 5 assumes that in a certain triangle ABC *the side* AB = *the side* AC this is the *hypothesis* From this it is required to prove that *the angle* ABC = *the angle* ACB this is the *conclusion*

If we interchange the hypothesis and conclusion of a theorem, we enunciate a new theorem which is called the **converse** of the first

For example, in Theorem 5

it is *assumed* that AB = AC,
it is *required to prove* that the angle ABC = the angle ACB

Now in Theorem 6

it is *assumed* that the angle ABC = the angle ACB,
it is *required to prove* that AB = AC

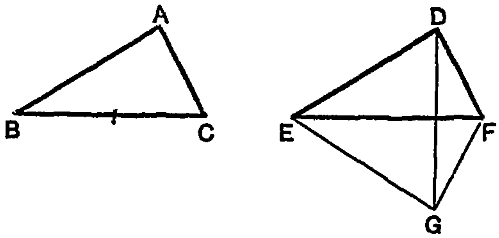
Thus we see that Theorem 6 is the converse of Theorem 5, (for the *hypothesis of each is the conclusion of the other*)

In Theorem 6 we employ an *indirect method of proof* frequently used in geometry It consists in shewing that the theorem *cannot be untrue*, since, if it were, we should be led to some *impossible conclusion* This form of proof is known as **Reductio ad Absurdum**, and is most commonly used in demonstrating the converse of some foregoing theorem

It must not however be supposed that if a theorem is true, its converse is *necessarily* true [See p 25]

THEOREM 7 [Euclid I 8]

If two triangles have the three sides of the one equal to the three sides of the other, each to each, they are equal in all respects



Let ABC, DEF be two triangles in which

AB = DE,
AC = DF,
BC = EF

It is required to prove that the triangles are equal in all respects

Proof Apply the △ABC to the △DEF,
so that B falls on E, and BC along EF, and
so that A is on the side of EF opposite to D
Then because BC = EF, C must fall on F

Let GEF be the new position of the △ABC
Join DG

Because ED = EG,
the ∠EDG = the ∠EGD *Theor* 5

Again, because FD = FG,
the ∠FDG = the ∠FGD

Hence the whole ∠EDF = the whole ∠EGF,
that is, the ∠EDF = the ∠BAC

Then in the △s BAC, EDF,

because { BA = ED,
AC = DF,
and the included ∠BAC = the included ∠EDF,

the triangles are equal in all respects *Theor* 4

Q E D

Obs. In this Theorem
it is *given* that AB = DE, BC = EF, CA = FD,
and we *prove* that ∠C = ∠F, ∠A = ∠D, ∠B = ∠E.
Also the triangles are equal in area.

Notice that the angles which are proved equal in the two triangles are opposite to sides which were given equal.

NOTE 1. We have taken the case in which DG falls within the ∠s EDF, EGF.

Two other cases might arise:

(i) DG might fall outside the ∠s EDF, EGF [as in Fig. 1].

(ii) DG might coincide with DF, FG [as in Fig. 2].

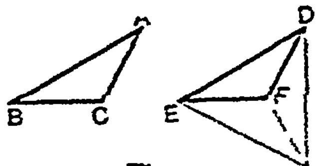


Fig. 1.

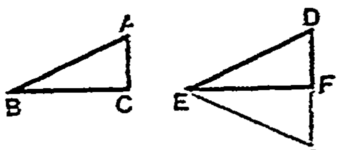


Fig. 2.

These cases will arise only when the given triangles are obtuse-angled or right-angled; and (as will be seen hereafter) not even then if we begin by choosing for superposition the *greatest* side of the △ ABC as in the diagram of page 24.

NOTE 2. Two triangles are said to be equiangular to one another when the angles of one are respectively equal to the angles of the other.

Hence if two triangles have the three sides of one severally equal to the three sides of the other, the triangles are equiangular to one another.

The student should state the converse theorem, and shew by a diagram that the converse is not necessarily true.

*** *At this stage Problems 1–3 and 8 [see page 70] may conveniently be taken, the proofs affording good illustrations of the Identical Equality of Two Triangles.*

EXERCISES

On The Identical Equality of Two Triangles
Theorems 4 and 7

(Theoretical)

1 Shew that the straight line which joins the vertex of an isosceles triangle to the middle point of the base,

(i) bisects the vertical angle

(ii) is perpendicular to the base

2 If ABCD is a rhombus, that is, an equilateral foursided figure, shew, by drawing the diagonal AC, that

(i) the angle ABC = the angle ADC,

(ii) AC bisects each of the angles BAD, BCD

3 If in a quadrilateral ABCD the opposite sides are equal, namely AB = CD and AD = CB, prove that the angle ADC = the angle ABC

4 If ABC and DBC are two isosceles triangles drawn on the same base BC, prove (by means of Theorem 7) that the angle ABD = the angle ACD, taking (i) the case where the triangles are on the *same* side of BC, (ii) the case where they are on *opposite* sides of BC

5 If ABC, DBC are two isosceles triangles drawn on opposite sides of the same base BC, and if AD be joined, prove that each of the angles BAC, BDC will be divided into two equal parts

6 Shew that the straight lines which join the extremities of the base of an isosceles triangle to the middle points of the opposite sides, are equal to one another

7 Two given points in the base of an isosceles triangle are equidistant from the extremities of the base shew that they are also equidistant from the vertex

8 Shew that the triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral

9 ABC is an isosceles triangle having AB equal to AC, and the angles at B and C are bisected by BO and CO shew that

(i) BO = CO,

(ii) AO bisects the angle BAC

10 Shew that the diagonals of a rhombus [see Ex 2] bisect one another at right angles

11 The equal sides BA, CA of an isosceles triangle BAC are produced beyond the vertex A to the points E and F, so that AE is equal to AF, and FB, EC are joined shew that FB is equal to EC

EXERCISES ON TRIANGLES

(Numerical and Graphical.)

1. Draw a triangle ABC, having given $a=2{\cdot}0''$, $b=2{\cdot}1''$, $c=1{\cdot}3''$. Measure the angles, and find their sum.

2. In the triangle ABC, $a=7{\cdot}5$ cm., $b=7{\cdot}0$ cm., and $c=6{\cdot}5$ cm. Draw and measure the perpendicular from B on CA.

3. Draw a triangle ABC, in which $a=7$ cm., $b=6$ cm., $C=65°$. How would you prove theoretically that any two triangles having these parts are alike in size and shape? Invent some experimental illustration.

4. Draw a triangle from the following data: $b=2''$, $c=2{\cdot}5''$, $A=57°$, and measure a, B, and C.

Draw a second triangle, using as data the values just found for a, B, and C; and measure b, c, and A. What conclusion do you draw?

5. A ladder, whose foot is placed 12 feet from the base of a house, reaches to a window 35 feet above the ground. Draw a plan in which 1" represents 10 ft., and find by measurement the length of the ladder.

6. I go due North 90 metres, then due East 20 metres. Plot my course (scale 1 cm. to 10 metres), and find by measurement as nearly as you can how far I am from my starting point.

7. When the sun is 12° above the horizon, a vertical pole casts a shadow 30 ft. long. Represent this on a diagram (scale 1" to 10 ft.) and find by measurement the approximate height of the pole.

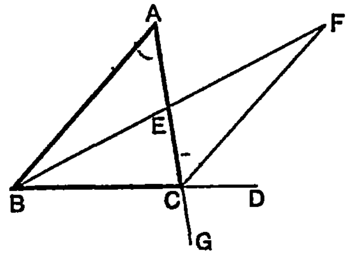
8. From a point A a surveyor goes 150 yards due East to B; then 300 yards due North to C; finally 450 yards due West to D. Plot his course (scale 1" to 100 yards), and find roughly how far D is from A. Measure the angle DAB, and say in what direction D bears from A.

9. B and C are two points known to be 260 yards apart, on a straight shore. A is a vessel at anchor. The angles CBA, BCA are observed to be 33° and 81° respectively. Find graphically the approximate distance of the vessel from the points B and C, and from the nearest point on shore.

10. In surveying a park it is required to find the distance between two points A and B; but as a lake intervenes, a direct measurement cannot be made. The surveyor therefore takes a third point C, from which both A and B are accessible, and he finds CA=245 yards, CB=320 yards, and the angle ACB=42°. Ascertain from a plan the approximate distance between A and B.

THEOREM 8 [Euclid I 16]

If one side of a triangle is produced, then the exterior angle is greater than either of the interior opposite angles



Let ABC be a triangle, and let BC be produced to D

It is required to prove that the exterior $\angle$ ACD *is greater than either of the interior opposite* $\angle^s$ ABC, BAC

Suppose E to be the middle point of AC
Join BE, and produce it to F, making EF equal to BE
Join FC

Proof Then in the $\triangle^s$ AEB, CEF,

because $\begin{cases} AE = CE, \\ EB = EF, \\ \text{and the } \angle AEB = \text{the vertically opposite } \angle CEF, \end{cases}$

the triangles are equal in all respects, *Theor* 4
so that the $\angle$ BAE = the $\angle$ ECF

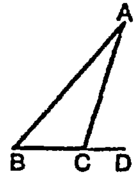
But the $\angle$ ECD is greater than the $\angle$ ECF,
the $\angle$ ECD is greater than the $\angle$ BAE,
that is, the $\angle$ ACD is greater than the $\angle$ BAC

In the same way, if AC is produced to G, by supposing A to be joined to the middle point of BC, it may be proved that the $\angle$ BCG is greater than the $\angle$ ABC

But the $\angle$ BCG = the vertically opposite $\angle$ ACD
the $\angle$ ACD is greater than the $\angle$ ABC

Q E D

COROLLARY 1 *Any two angles of a triangle are together less than two right angles*

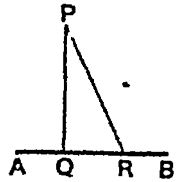


For the ∠ABC is less than the ∠ACD *Proved*
to each add the ∠ACB
Then the ∠s ABC, ACB are less than the ∠s ACD, ACB,
therefore less than two right angles

COROLLARY 2 *Every triangle must have at least two acute angles*

For if one angle is obtuse or a right angle, then by Cor 1 each of the other angles must be less than a right angle

COROLLARY 3 *Only one perpendicular can be drawn to a straight line from a given point outside it*



If two perpendiculars could be drawn to AB from P, we should have a triangle PQR in which each of the ∠s PQR, PRQ would be a right angle, which is impossible.

EXERCISES

1 Prove Corollary 1 by joining the vertex A to any point in the base BC

2 ABC is a triangle and D any point within it *If* BD *and* CD *are joined, the angle* BDC *is greater than the angle* BAC Prove this

(i) by producing BD to meet AC

(ii) by joining AD, and producing it towards the base

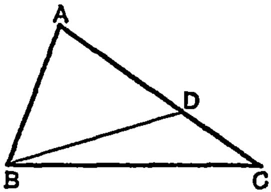
3 If any side of a triangle is produced both ways, the exterior angles so formed are together greater than two right angles

4 To a given straight line there cannot be drawn from a point outside it more than two straight lines of the same given length

5 If the equal sides of an isosceles triangle are produced, the exterior angles must be obtuse

THEOREM 9 [Euclid I. 18]

If one side of a triangle is greater than another, then the angle opposite to the greater side is greater than the angle opposite to the less



Let ABC be a triangle, in which the side AC is greater than the side AB

It is required to prove that the ∠ABC *is greater than the* ∠ACB

From AC cut off AD equal to AB
Join BD

Proof Because AB = AD,
∴ the ∠ABD = the ∠ADB *Theor* 5

But the exterior ∠ADB of the △BDC is greater than the interior opposite ∠DCB, that is, greater than the ∠ACB

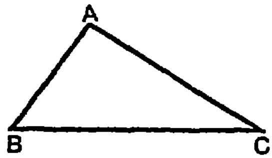
∴ the ∠ABD is greater than the ∠ACB
Still more then is the ∠ABC greater than the ∠ACB

Q E D

Obs The mode of demonstration used in the following Theorem is known as the Proof by Exhaustion. It is applicable to cases in which *one* of certain suppositions must necessarily be true, and it consists in shewing that each of these suppositions is false *with one exception* hence the truth of the remaining supposition is inferred

THEOREM 10 [Euclid I 19]

If one angle of a triangle is greater than another, then the side opposite to the greater angle is greater than the side opposite to the less



Let ABC be a triangle, in which the ∠ABC is greater than the ∠ACB

It is required to prove that the side AC *is greater than the side* AB

Proof If AC is not greater than AB,
it must be either equal to, or less than AB

Now if AC were equal to AB,
then the ∠ABC would be equal to the ∠ACB, *Theor* 5
but, by hypothesis, it is not

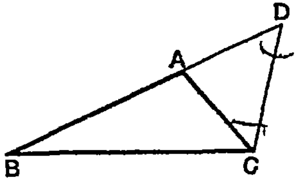
Again, if AC were less than AB,
then the ∠ABC would be less than the ∠ACB, *Theor* 9
but, by hypothesis, it is not

That is, AC is neither equal to, nor less than AB
AC is greater than AB Q E D

[For Exercises on Theorems 9 and 10 see page 34.]

THEOREM 11 [Euclid I 20]

Any two sides of a triangle are together greater than the third side



Let ABC be a triangle

It is required to prove that any two of its sides are together greater than the third side

It is enough to shew that if BC is the greatest side, then BA, AC are together greater than BC.

Produce BA to D, making AD equal to AC
Join DC

Proof. Because AD = AC,
the $\angle$ ACD = the $\angle$ ADC *Theor* 5

But the $\angle$ BCD is greater than the $\angle$ ACD,
the $\angle$ BCD is greater than the $\angle$ ADC,
that is, than the $\angle$ BDC

Hence from the $\triangle$ BDC,
BD is greater than BC *Theor* 10

But BD = BA and AC together,
BA and AC are together greater than BC

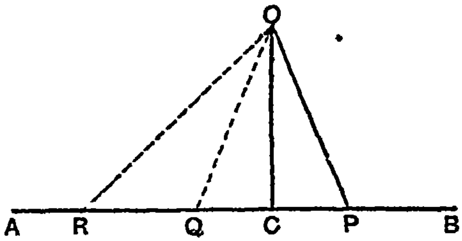
Q E D

NOTE This proof may serve as an exercise, but the truth of the Theorem is really self evident For to go from B to C along the straight line BC is clearly shorter than to go from B to A and then from A to C In other words

The shortest distance between two points is the straight line which joins them.

THEOREM 12

Of all straight lines drawn from a given point to a given straight line the perpendicular is the least



Let OC be the perpendicular, and OP *any* oblique, drawn from the given point O to the given straight line AB

It is required to prove that OC *is less than* OP

Proof In the △ OCP, since the ∠ OCP is a right angle,
the ∠ OPC is less than a right angle, *Theor* 8 *Cor*
that is, the ∠ OPC is less than the ∠ OCP
∴ OC is less than OP *Theor* 10.

Q E D

COROLLARY 1 Hence conversely, since there can be only one perpendicular and one shortest line from O to AB,

If OC *is the shortest straight line from* O *to* AB, *then* OC *is perpendicular to* AB

COROLLARY 2 *Two obliques* OP, OQ, *which cut* AB *at equal distances from* C *the foot of the perpendicular, are equal*

The △ˢ OCP, OCQ may be shewn to be congruent by Theorem 4,
∴ hence OP=OQ.

COROLLARY 3 *Of two obliques* OQ, OR, *if* OR *cuts* AB *at the greater distance from* C *the foot of the perpendicular, then* OR *is greater than* OQ.

The ∠OQC is acute, ∴ the ∠OQR is obtuse,
∴ the ∠OQR is greater than the ∠ORQ,
∴ OR is greater than OQ.

EXERCISES ON INEQUALITIES IN A TRIANGLE

1 *The hypotenuse is the greatest side of a right angled triangle.*

2 *The greatest side of any triangle makes acute angles with each of the other sides*

3 *If from the ends of a side of a triangle, two straight lines are drawn to a point within the triangle, then these straight lines are together less than the other two sides of the triangle*

4 BC, the base of an isosceles triangle ABC, is produced to any point D, shew that AD is greater than either of the equal sides

5 If in a quadrilateral the greatest and least sides are opposite to one another, then each of the angles adjacent to the least side is greater than its opposite angle

6 In a triangle ABC, if AC is not greater than AB, shew that any straight line drawn through the vertex A and terminated by the base BC, is less than AB

7 ABC is a triangle, in which OB, OC bisect the angles ABC, ACB respectively shew that, if AB is greater than AC, then OB is greater than OC

8 The difference of any two sides of a triangle is less than the third side

9 The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter

10 The perimeter of a quadrilateral is greater than the sum of its diagonals

11 ABC is a triangle, and the vertical angle BAC is bisected by a line which meets BC in X, shew that BA is greater than BX, and CA greater than CX Hence obtain a proof of Theorem 11

12 The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle

13 The sum of the diagonals of a quadrilateral is less than the sum of the four straight lines drawn from the angular points to any given point Prove this, and point out the exceptional case

14 *In a triangle any two sides are together greater than twice the median which bisects the remaining side*

[Produce the median, and complete the construction after the manner of Theorem 8]

15 *In any triangle the sum of the medians is less than the perimeter*

PARALLELS

DEFINITION Parallel straight lines are such as, being in the same plane, do not meet however far they are produced beyond both ends

NOTE. Parallel lines must be *in the same plane* For instance, two straight lines, one of which is drawn on a table and the other on the floor would never meet if produced, but they are not for that reason necessarily parallel

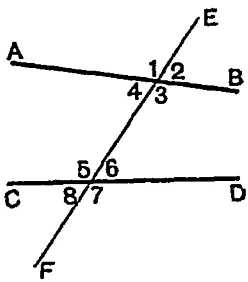
AXIOM. *Two intersecting straight lines cannot both be parallel to a third straight line*

In other words

Through a given point there can be only one straight line parallel to a given straight line

This assumption is known as *Playfair's Axiom*

DEFINITION When two straight lines AB, CD are met by a third straight line EF, *eight* angles are formed, to which for the sake of distinction particular names are given.



Thus in the adjoining figure,
1, 2, 7, 8 are called exterior angles,
3, 4, 5, 6 are called interior angles,
4 and 6 are said to be alternate angles, so also the angles 3 and 5 are alternate to one another

Of the angles 2 and 6, 2 is referred to as the exterior angle, and 6 as the interior opposite angle on the same side of EF Such angles are also known as corresponding angles Similarly 7 and 3, 8 and 4, 1 and 5 are pairs of corresponding angles

THEOREM 13 [Euclid I 27 and 28]

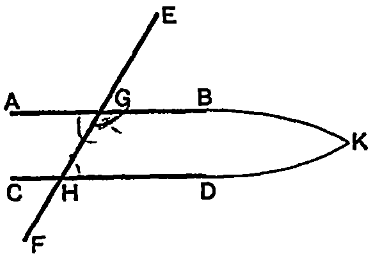
If a straight line cuts two other straight lines so as to make

(i) *the alternate angles equal,*

or (ii) *an exterior angle equal to the interior opposite angle on the same side of the cutting line,*

or (iii) *the interior angles on the same side equal to two right angles,*

then in each case the two straight lines are parallel



(i) Let the straight line EGHF cut the two straight lines AB, CD at G and H so as to make the alternate $\angle^s$ AGH, GHD equal to one another

It is required to prove that AB *and* CD *are parallel*

Proof If AB and CD are not parallel, they will meet, if produced, either towards B and D, or towards A and C

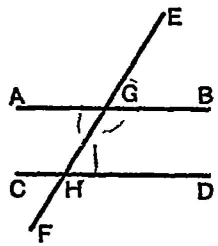
If possible, let AB and CD, when produced, meet towards B and D, at the point K

Then KGH is a triangle, of which one side KG is produced to A, the exterior $\angle$AGH is greater than the interior opposite $\angle$GHK, but, by hypothesis, it is not greater Th, 8 lt

AB and CD cannot meet when produced towards B and D

Similarly it may be shewn that they cannot meet towards A and C

AB and CD are parallel



(ii) Let the exterior $\angle$ EGB = the interior opposite $\angle$ GHD

It is required to prove that AB *and* CD *are parallel*

Proof Because the $\angle$ EGB = the $\angle$ GHD,
and the $\angle$ EGB = the vertically opposite $\angle$ AGH,
the $\angle$ AGH = the $\angle$ GHD

and these are alternate angles,
AB and CD are parallel

(iii) Let the two interior $\angle$s BGH, GHD be together equal to two right angles

It is required to prove that AB *and* CD *are parallel*

Proof Because the $\angle$s BGH, GHD together = two right angles,
and because the adjacent $\angle$s BGH, AGH together = two right angles,
the $\angle$s BGH, AGH together = the $\angle$s BGH, GHD

From these equals take the $\angle$ BGH,
then the remaining $\angle$ AGH = the remaining $\angle$ GHD

and these are alternate angles,
AB and CD are parallel

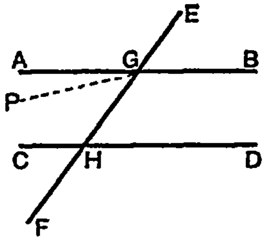
Q E D

DEFINITION A straight line drawn across a set of given lines is called a transversal

For instance, in the above diagram the line EGHF, which crosses the given lines AB, CD is a transversal

Theorem 14 [Euclid I 29]

If a straight line cuts two parallel lines, it makes
(i) *the alternate angles equal to one another,*
(ii) *the exterior angle equal to the interior opposite angle on the same side of the cutting line,*
(iii) *the two interior angles on the same side together equal to two right angles*



Let the straight lines AB, CD be parallel, and let the straight line EGHF cut them.

It is required to prove that
(i) *the* ∠AGH = *the alternate* ∠GHD,
(ii) *the exterior* ∠EGB = *the interior opposite* ∠GHD,
(iii) *the two interior* ∠ˢ BGH, GHD *together* = *two right angles*

Proof (i) If the ∠AGH is not equal to the ∠GHD, suppose the ∠PGH equal to the ∠GHD, and alternate to it,
then PG and CD are parallel *Theor* 13

But, by hypothesis, AB and CD are parallel,
the two intersecting straight lines AG, PG are both parallel to CD which is impossible. *Playfair's Axiom*

the ∠AGH is not unequal to the ∠GHD,
that is, the alternate ∠ˢ AGH, GHD are equal

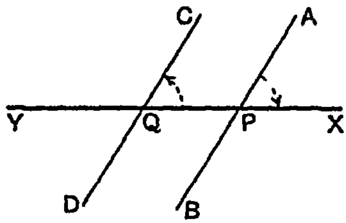
(ii) Again, because the ∠EGB = the vertically opposite ∠AGH,
and the ∠AGH = the alternate ∠GHD, *Proved*
the exterior ∠EGB = the interior opposite ∠GHD

(iii) Lastly, the $\angle$ EGB = the $\angle$ GHD, *Proved*
add to each the $\angle$ BGH,
then the $\angle$s EGB, BGH together = the angles BGH, GHD
But the adjacent $\angle$s EGB, BGH together = two right angles,
the two interior $\angle$s BGH, GHD together = two right angles
Q E D

PARALLELS ILLUSTRATED BY ROTATION

The direction of a straight line is determined by the angle which it makes with some given line of reference

Thus the direction of AB, relatively to the given line YX, is given by the angle APX



Now suppose that AB and CD in the adjoining diagram are parallel, then we have learned that the ext $\angle$APX = the int opp $\angle$CQX, that is, AB and CD *make equal angles* with the line of reference YX

This brings us to the leading idea connected with parallels *Parallel straight lines have the same* DIRECTION, *but differ in* POSITION

The same idea may be illustrated thus

Suppose AB to rotate about P through the $\angle$APX, so as to take the position XY Thence let it rotate about Q *the opposite way* through the equal $\angle$XQC it will now take the position CD Thus AB may be brought into the position of CD by two rotations which, being equal and opposite, involve no final change of direction

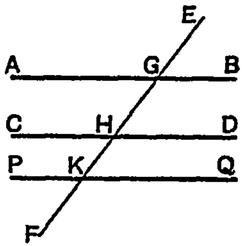
HYPOTHETICAL CONSTRUCTION In the above diagram let AB be a fixed straight line, Q a fixed point, CD a straight line turning about Q, and YQPX any transversal through Q Then as CD rotates, there must be one position in which the $\angle$CQX = the fixed $\angle$APX

Hence through any given point we may assume a line to pass parallel to any given straight line

Obs If AB is a straight line, movements from A towards B, and from B towards A are said to be in opposite senses of the line AB

Theorem 15 [Euclid I. 30]

Straight lines which are parallel to the same straight line are parallel to one another



Let the straight lines AB, CD be each parallel to the straight line PQ.

It is required to prove that AB *and* CD *are parallel to one another*

Draw a straight line EF cutting AB, CD, and PQ in the points G, H, and K

Proof Then because AB and PQ are parallel, and EF meets them,

the $\angle$AGK = the alternate $\angle$GKQ.

And because CD and PQ are parallel, and EF meets them,

the exterior $\angle$GHD = the interior opposite $\angle$GKQ

the $\angle$AGH = the $\angle$GHD,

and these are alternate angles,

AB and CD are parallel

Q E D

Note. If PQ lies between AB and CD, the Proposition needs no proof, for it is inconceivable that two straight lines, which do not meet an intermediate straight line, should meet one another

The truth of this Proposition may be readily deduced from Playfair's Axiom, of which it is the converse

For if AB and CD were not parallel, they would meet when produced Then there would be two intersecting straight lines both parallel to a third straight line which is impossible

Therefore AB and CD never meet, that is, they are parallel

EXERCISES ON PARALLELS

1 In the diagram of the previous page, if the angle EGB is 55°, express in degrees each of the angles GHC, HKQ, QKF

2 *Straight lines which are perpendicular to the same straight line are parallel to one another*

3 *If a straight line meet two or more parallel straight lines, and is perpendicular to one of them, it is also perpendicular to all the others*

4 *Angles of which the arms are parallel, each to each, are either equal or supplementary*

5 Two straight lines AB, CD bisect one another at O Shew that the straight lines joining AC and BD are parallel

6 Any straight line drawn parallel to the base of an isosceles triangle makes equal angles with the sides

7 If from any point in the bisector of an angle a straight line is drawn parallel to either arm of the angle, the triangle thus formed is isosceles

8 From X, a point in the base BC of an isosceles triangle ABC, a straight line is drawn at right angles to the base, cutting AB in Y, and CA produced in Z shew the triangle AYZ is isosceles

9 If the straight line which bisects an exterior angle of a triangle is parallel to the opposite side, shew that the triangle is isosceles

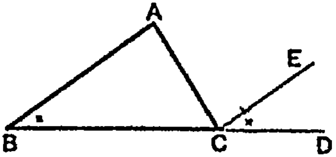
10 The straight lines drawn from any point in the bisector of an angle parallel to the arms of the angle, and terminated by them, are equal and the resulting figure is a rhombus

11 AB and CD are two straight lines intersecting at D, and the adjacent angles so formed are bisected if through any point X in DC a straight line YXZ is drawn parallel to AB and meeting the bisectors in Y and Z, shew that XY is equal to XZ

12 Two straight rods PA, QB revolve about pivots at P and Q, PA making 12 complete revolutions a minute, and QB making 10 If they start parallel and pointing the same way, how long will it be before they are again parallel, (i) pointing opposite ways, (ii) pointing the same way?

THEOREM 16. [Euclid I. 32.]

The three angles of a triangle are together equal to two right angles.



Let ABC be a triangle.

It is required to prove that the three $\angle$s ABC, BCA, CAB *together* = *two right angles.*

Produce BC to any point D; and suppose CE to be the line through C parallel to BA.

Proof. Because BA and CE are parallel and AC meets them,
the $\angle$ACE = the alternate $\angle$CAB.

Again, because BA and CE are parallel, and BD meets them,
the exterior $\angle$ECD = the interior opposite $\angle$ABC.

∴ *the whole exterior* $\angle$ACD = *the sum of the two interior opposite* $\angle$s CAB, ABC.

To each of these equals add the $\angle$BCA:
then the $\angle$s BCA, ACD together = the three $\angle$s BCA, CAB, ABC.

But the adjacent $\angle$s BCA, ACD together = two right angles.
∴ the $\angle$s BCA, CAB, ABC together = two right angles.

Q.E.D.

Obs. In the course of this proof the following most important property has been established.

If a side of a triangle is produced the exterior angle is equal to the sum of the two interior opposite angles.

Namely, the ext. $\angle$ACD = the $\angle$CAB + the $\angle$ABC.

INFERENCES FROM THEOREM 16

√ 1 *If* A, B, *and* C *denote the number of degrees in the angles of a triangle,*

$$\textit{then } A + B + C = 180°$$

√ 2 *If two triangles have two angles of the one respectively equal to two angles of the other, then the third angle of the one is equal to the third angle of the other*

√ 3 *In any right-angled triangle the two acute angles are complementary*

√ 4 *If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled*

√ 5 *The sum of the angles of any quadrilateral figure is equal to four right angles*

EXERCISES ON THEOREM 16

√ 1 Each angle of an equilateral triangle is two thirds of a right angle, or 60°

√ 2. In a right-angled isosceles triangle each of the equal angles is 45°

√ 3 Two angles of a triangle are 36° and 123° respectively deduce the third angle, and verify your result by measurement

√ 4 In a triangle ABC, the $\angle B = 111°$, the $\angle C = 42°$, deduce the $\angle A$, and verify by measurement.

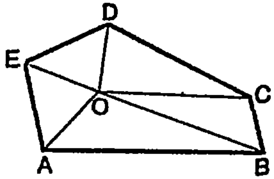
5 One side BC of a triangle ABC is produced to D If the exterior angle ACD is 134°, and the angle BAC is 42°, find each of the remaining interior angles

6 In the figure of Theorem 16, if the $\angle ACD = 118°$, and the $\angle B = 51°$, find the $\angle$s A and C, and check your results by measurement

7 *Prove that the three angles of a triangle are together equal to two right angles by supposing a line drawn through the vertex parallel to the base*

8 *If two straight lines are perpendicular to two other straight lines, each to each, the acute angle between the first pair is equal to the acute angle between the second pair*

COROLLARY 1 *All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides*



Let ABCDE be a rectilineal figure of n sides

It is required to prove that all the interior angles $+ 4$ rt $\angle^s$ $= 2n$ rt $\angle^s$

Take any point O within the figure, and join O to each of its vertices

Then the figure is divided into n triangles

And the three $\angle^s$ of each $\triangle$ together $= 2$ rt $\angle^s$
Hence *all* the $\angle^s$ of *all* the $\triangle^s$ together $= 2n$ rt $\angle^s$

But all the $\angle^s$ of all the $\triangle^s$ make up all the interior angles of the figure together with the angles at O, which $= 4$ rt $\angle^s$

all the int $\angle^s$ of the figure $+ 4$ rt $\angle^s = 2n$ rt $\angle^s$

Q E D

DEFINITION A regular polygon is one which has all its sides equal and all its angles equal

Thus if D denotes the number of degrees in each angle of a *regular* polygon of n sides, the above result may be stated thus

$$n\mathrm{D} + 360° = n \quad 180°$$

EXAMPLE

Find the number of degrees in each angle of

(i) a regular hexagon (6 sides),
(ii) a regular octagon (8 sides),
(iii) a regular decagon (10 sides)

EXERCISES ON THEOREM 16

(*Numerical and Graphical*)

1 ABC is a triangle in which the angles at B and C are respectively double and treble of the angle at A find the number of degrees in each of these angles

2 Express in degrees the angles of an isosceles triangle in which

(i) Each base angle is double of the vertical angle,

(ii) Each base angle is four times the vertical angle

3 The base of a triangle is produced both ways, and the exterior angles are found to be 94° and 126°, deduce the vertical angle Construct such a triangle, and check your result by measurement

4 The sum of the angles at the base of a triangle is 162°, and their difference is 60° find all the angles

5 The angles at the base of a triangle are 84° and 62°, deduce (i) the vertical angle (ii) the angle between the bisectors of the base angles Check your results by construction and measurement

6 In a triangle ABC, the angles at B and C are 74° and 62°, if AB and AC are produced, deduce the angle between the bisectors of the exterior angles Check your result graphically

7 Three angles of a quadrilateral are respectively 114½°, 50°, and 75½°, find the fourth angle

8 In a quadrilateral ABCD the angles at B, C, and D are respectively equal to 2A, 3A, and 4A, find all the angles

9 Four angles of an irregular pentagon (5 sides) are 40°, 78°, 122°, and 135°, find the fifth angle

10 In any *regular* polygon of n sides, each angle contains $\frac{2(n-2)}{n}$ right angles

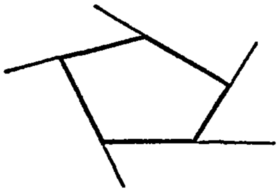
(i) Deduce this result from the Enunciation of Corollary 1

(ii) Prove it independently by joining one vertex A to each of the others (except the two immediately adjacent to A), thus dividing the polygon into $n-2$ triangles

11 How many sides have the regular polygons each of whose angles is (i) 108°, (ii) 156°?

12 Shew that the only *regular* figures which may be fitted together so as to form a plane surface are (i) *equilateral triangles,* (ii) *squares,* (iii) *regular hexagons*

Corollary 2 *If the sides of a rectilineal figure, which has no re-entrant angle, are produced in order, then all the exterior angles so formed are together equal to four right angles.*



1st Proof. Suppose, as before, that the figure has n sides, and consequently n vertices

Now at *each* vertex

the interior $\angle$ + the exterior $\angle = 2$ rt $\angle^s$,

and there are n vertices,

the sum of the int $\angle^s$ + the sum of the ext $\angle^s = 2n$ rt $\angle^s$

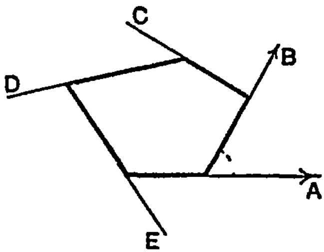
But by Corollary 1,

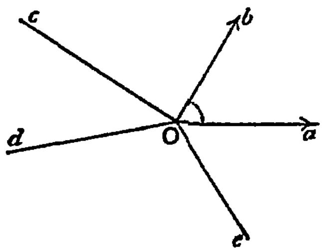
the sum of the int $\angle^s$ + 4 rt $\angle^s$ $= 2n$ rt $\angle^s$,

the sum of the ext $\angle^s = 4$ rt $\angle^s$

Q E D

2nd Proof





Take any point O, and suppose Oa, Ob, Oc, Od, and Oe, are lines parallel to the sides marked, A, B, C, D, E (and drawn from O in the *sense* in which those sides were produced)

Then the exterior $\angle$ between the sides A and B = the $\angle\, a$Ob

And the other exterior $\angle^s$ = the $\angle^s$ bOc, cOd, dOe, eOa, respectively

the sum of the ext $\angle^s$ = the sum of the $\angle^s$ at O

$= 4$ rt. $\angle^s$

EXERCISES

1 If one side of a regular hexagon is produced, shew that the exterior angle is equal to the interior angle of an equilateral triangle.

2 Express in degrees the magnitude of each *exterior* angle of (i) a regular octagon, (ii) a regular decagon

3 How many sides has a regular polygon if each exterior angle is (i) 30°, (ii) 24°?

4 If a straight line meets two parallel straight lines, and the two interior angles on the same side are bisected, shew that the bisectors meet at right angles

5 If the base of any triangle is produced both ways, shew that the sum of the two exterior angles *minus* the vertical angle is equal to two right angles

6 In the triangle ABC the base angles at B and C are bisected by BO and CO respectively Shew that the angle $BOC = 90° + \frac{A}{2}$

7 In the triangle ABC, the sides AB, AC are produced, and the exterior angles are bisected by BO and CO Shew that the angle $BOC = 90° - \frac{A}{2}$

8 The angle contained by the bisectors of two adjacent angles of a quadrilateral is equal to half the sum of the remaining angles

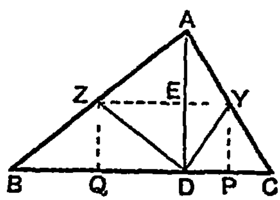
9 A is the vertex of an isosceles triangle ABC, and BA is produced to D, so that AD is equal to BA, if DC is drawn, shew that BCD is a right angle

10 The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

EXPERIMENTAL PROOF OF THEOREM 16 [$A+B+C=180°$]

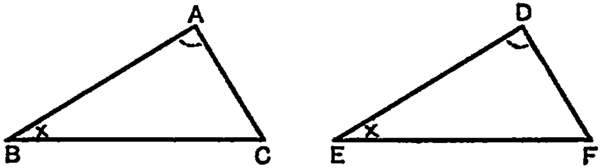
In the △ ABC, AD is perp to BC the greatest side AD is bisected at right angles by ZY, and YP, ZQ are perp[s] on BC

If now the △ is folded about the three dotted lines, the ∠s A, B, and C will coincide with the ∠s ZDY, ZDQ, YDP, their sum is 180°



Theorem 17 [Euclid I. 26]

If two triangles have two angles of one equal to two angles of the other, each to each, and any side of the first equal to the corresponding side of the other, the triangles are equal in all respects



Let ABC, DEF be two triangles in which

the $\angle A =$ the $\angle D$,

the $\angle B =$ the $\angle E$,

also let the side $BC =$ the corresponding side EF

It is required to prove that the $\triangle^s$ ABC, DEF *are equal in all respects*

Proof The sum of the $\angle^s$ A, B, and C

$= 2$ rt $\angle^s$ *Theor* 16

$=$ the sum of the $\angle^s$ D, E, and F,

and the $\angle^s$ A and $B =$ the $\angle^s$ D and E respectively,

the $\angle C =$ the $\angle F$

Apply the $\triangle$ ABC to the $\triangle$ DEF, so that B falls on E, and BC along EF

Then because $BC = EF$,

C must coincide with F

And because the $\angle B =$ the $\angle E$,

BA must fall along ED

And because the $\angle C =$ the $\angle F$,

CA must fall along FD

the point A, which falls both on ED and on FD, must coincide with D, the point in which these lines intersect

the $\triangle$ ABC coincides with the $\triangle$ DEF,

and is therefore equal to it in all respects

So that $AB = DE$, and $AC = DF$,

and the $\triangle ABC =$ the $\triangle$ DEF in area. Q E D

EXERCISES

ON THE IDENTICAL EQUALITY OF TRIANGLES

1 Shew that the perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides are equal

2 *Any point on the bisector of an angle is equidistant from the arms of the angle*

3 Through O, the middle point of a straight line AB, any straight line is drawn, and perpendiculars AX and BY are dropped upon it from A and B shew that AX is equal to BY

4 If the bisector of the vertical angle of a triangle is at right angles to the base, the triangle is isosceles

5 If in a triangle the perpendicular from the vertex on the base bisects the base, then the triangle is isosceles

6 If the bisector of the vertical angle of a triangle also bisects the base, the triangle is isosceles

[Produce the bisector, and complete the construction after the manner of Theorem 8]

7 The middle point of any straight line which meets two parallel straight lines, and is terminated by them, is equidistant from the parallels

8 A straight line drawn between two parallels and terminated by them, is bisected, shew that any other straight line passing through the middle point and terminated by the parallels, is also bisected at that point

9 If through a point equidistant from two parallel straight lines, two straight lines are drawn cutting the parallels, the portions of the latter thus intercepted are equal

10 In a quadrilateral, ABCD, if AB=AD, and BC=DC shew that the diagonal AC bisects each of the angles which it joins, and that AC is perpendicular to BD

11 *A surveyor wishes to ascertain the breadth of a river which he cannot cross Standing at a point* A *near the bank, he notes an object* B *immediately opposite on the other bank He lays down a line* AC *of any length at right angles to* AB, *fixing a mark at* O *the middle point of* AC *From* C *he walks along a line perpendicular to* AC *until he reaches a point* D *from which* O *and* B *are seen in the same direction He now measures* CD *prove that the result gives him the width of the river*

ON THE IDENTICAL EQUALITY OF TRIANGLES

Three cases of the congruence of triangles have been dealt with in Theorems 4, 7, 17, the results of which may be summarised as follows

Two triangles are equal in all respects when the following three parts in each are severally equal

1 *Two sides, and the included angle* — *Theorem* 4

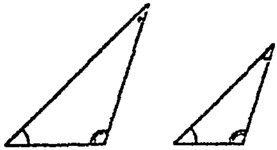
2 *The three sides* — *Theorem* 7

3 *Two angles and one side, the side given in one triangle* CORRESPONDING *to that given in the other* — *Theorem* 17

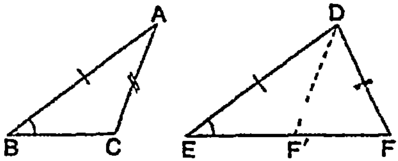
Two triangles are not, however, necessarily equal in all respects when *any three parts* of one are equal to the corresponding parts of the other

For example

(i) When the *three angles* of one are equal to the *three angles* of the other, each to each, the adjoining diagram shews that the triangles need not be equal in all respects



(ii) When *two sides and one angle* in one are equal to *two sides and one angle* of the other, the given angles being *opposite* to equal sides, the diagram below shews that the triangles need not be equal in all respects



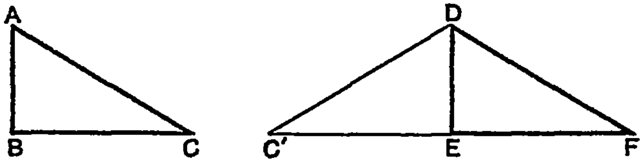
For if AB = DE, and AC = DF, and the ∠ABC = the ∠DEF, it will be seen that the shorter of the given sides in the triangle DEF may lie in either of the positions DF or DF′

NOTE From these data it may be shewn that the angles opposite to the equal sides AB, DE are either *equal* (as for instance the ∠s ACB, DF′E) or *supplementary* (as the ∠s ACB, DFE), and that in the former case the triangles are equal in all respects This is called the ambiguous case in the congruence of triangles [See Problem 9, p 82]

If the given angles at B and E are right angles, the ambiguity disappears This exception is proved in the following Theorem

THEOREM 18

Two right-angled triangles which have their hypotenuses equal, and one side of one equal to one side of the other, are equal in all respects



Let ABC, DEF be two right-angled triangles, in which
the ∠ˢ ABC, DEF are right angles,
the hypotenuse AC = the hypotenuse DF,
and AB = DE

It is required to prove that the △ˢ ABC, DEF *are equal in all respects*

Proof Apply the △ABC to the △DEF, so that AB falls on the equal line DE, and C on the side of DE opposite to F

Let C′ be the point on which C falls.
Then DEC′ represents the △ABC in its new position

Since each of the ∠ˢ DEF, DEC′ is a right angle,
EF and EC′ are in one straight line

And in the △C′DF, because DF = DC′ (*i e* AC),
∴ the ∠DFC = the ∠DC′F *Theor* 5

Hence in the △ˢ DEF, DEC′,

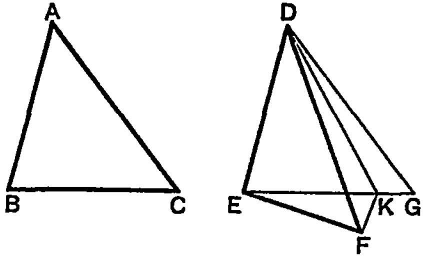
because the ∠DEF = the ∠DEC′, being right angles,
the ∠DFE = the ∠DC′E, *Proved*
and the side DE is common

the △ˢ DEF, DEC′ are equal in all respects, *Theor* 17.
that is, the △ˢ DEF, ABC are equal in all respects

Q E.D

THEOREM 19 [Euclid I 24]

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle included by the two sides of one greater than the angle included by the corresponding sides of the other, then the base of that which has the greater angle is greater than the base of the other



Let ABC, DEF be two triangles, in which

BA = ED,

and AC = DF,

but the ∠ BAC is greater than the ∠ EDF

It is required to prove that the base BC *is greater than the base* EF

Proof Apply the △ ABC to the △ DEF, so that A falls on D, and AB along DE

Then because AB = DE, B must coincide with E

Let DG, GE represent AC, CB in their new position

Then if EG passes through F, EG is greater than EF,

that is, BC is greater than EF

But if EG does not pass through F, suppose that DK bisects the ∠ FDG, and meets EG in K Join FK

Then in the △ˢ FDK, GDK,

because $\begin{cases} \text{FD} = \text{GD}, \\ \text{DK is common to both}, \\ \text{and the included } \angle\, \text{FDK} = \text{the included } \angle\, \text{GDK}, \end{cases}$

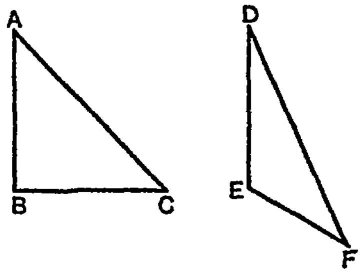
FK = GK *Theor* 4.

Now the two sides EK, KF are greater than EF,

that is, EK, KG are greater than EF

EG (or BC) is greater than EF Q.E D

Conversely, *if two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other, then the angle contained by the sides of that which has the greater base, is greater than the angle contained by the corresponding sides of the other*



Let ABC, DEF be two triangles in which

BA = ED,

and AC = DF,

but the base BC is greater than the base EF

It is required to prove that the ∠ BAC *is greater than the* ∠ EDF.

Proof If the ∠ BAC is not greater than the ∠ EDF,
it must be either equal to, or less than the ∠ EDF

Now if the ∠ BAC were equal to the ∠ EDF,
then the base BC would be equal to the base EF *Theor* 4
but, by hypothesis, it is not.

Again, if the ∠ BAC were less than the ∠ EDF,
then the base BC would be less than the base EF, *Theor* 19
but, by hypothesis, it is not

That is, the ∠ BAC is neither equal to, nor less than the ∠ EDF,
the ∠ BAC is greater than the ∠ EDF

Q E D

* *Theorems marked with an asterisk may be omitted or postponed at the discretion of the teacher*

REVISION LESSON ON TRIANGLES

1 State the properties of a triangle relating to

(i) the sum of its interior angles,

(ii) the sum of its exterior angles

What property corresponds to (i) in a polygon of n sides? With what other figures does a triangle share the property (ii)?

2 Classify triangles with regard to their angles Enunciate any Theorem or Corollary assumed in the classification

3 Enunciate two Theorems in which from data relating to the sides a conclusion is drawn relating to the angles

In the triangle ABC, if $a=3{\cdot}6$ cm, $b=2{\cdot}8$ cm, $c=3{\cdot}6$ cm, arrange the angles in order of their sizes (before measurement), and prove that the triangle is acute-angled.

4 Enunciate two Theorems in which from data relating to the angles a conclusion is drawn relating to the sides

In the triangle ABC, if

(i) $A=48^\circ$ and $B=51^\circ$, find the third angle, and name the greatest side

(ii) $A=B=62\frac{1}{3}^\circ$, find the third angle, and arrange the sides in order of their lengths

5 From which of the conditions given below may we conclude that the triangles ABC, A′B′C′ are identically equal? Point out where ambiguity arises, and draw the triangle ABC in each case

(i) $\begin{cases} A=A'=71^\circ \\ B=B\ =46^\circ \\ a=a'=3{\cdot}7 \text{ cm} \end{cases}$ (ii) $\begin{cases} a=a'=4{\cdot}2 \text{ cm} \\ b=b'=2{\cdot}4 \text{ cm} \\ C=C\ =81^\circ \end{cases}$ (iii) $\begin{cases} A=A'=36^\circ \\ B=B'=121^\circ \\ C=C'=23^\circ \end{cases}$

(iv) $\begin{cases} a=a\ =3{\cdot}0 \text{ cm} \\ b=b'=5{\cdot}2 \text{ cm} \\ c=c'=4{\cdot}5 \text{ cm} \end{cases}$ (v) $\begin{cases} B=B'=53^\circ \\ b=b'=4{\cdot}3 \text{ cm.} \\ c=c'=5{\cdot}0 \text{ cm} \end{cases}$ (vi) $\begin{cases} C=C\ =90^\circ \\ c=c'=5 \text{ cm} \\ a=a'=3 \text{ cm} \end{cases}$

6 Summarise the results of the last question by stating generally under what conditions two triangles

(i) are necessarily congruent,

(ii) may or may not be congruent.

7 *If two triangles have their angles equal, each to each, the triangles are not necessarily equal in all respects, because the three data are not independent.* Carefully explain this statement.

(Miscellaneous Examples.)

8. (i) *The perpendicular is the shortest line that can be drawn to a given straight line from a given point.*

(ii) **Obliques** *which make equal angles with the perpendicular are equal.*

(iii) *Of two obliques the less is that which makes the smaller angle with the perpendicular.*

9. *If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides are either equal or supplementary, and in the former case the triangles are equal in all respects.*

10. PQ is a perpendicular (4 cm. in length) to a straight line XY. Draw through P a series of obliques making with PQ the angles 15°, 30°, 45°, 60°, 75°. *Measure* the lengths of these obliques, and tabulate the results.

11. PAB is a triangle in which AB and AP have constant lengths 4 cm. and 3 cm. If AB is fixed, and AP rotates about A, trace the changes in PB, as the angle A increases from 0° to 180°.

Answer this question by drawing a series of figures, increasing A by increments of 30°. Measure PB in each case, and tabulate the results.

12. From B the foot of a flagstaff AB a horizontal line is drawn passing two points C and D which are 27 feet apart. The angles BCA and BDA are 65° and 40° respectively. Represent this on a diagram (scale 1 cm. to 10 ft.), and find by measurement the approximate height of the flagstaff.

13. From P, the top of a lighthouse PQ, two boats A and B are seen at anchor in a line due south of the lighthouse. It is known that PQ=126 ft., ∠PAQ=57°, ∠PBQ=33°; hence draw a plan in which 1" represents 100 ft., and find by measurement the distance between A and B to the nearest foot.

14. From a lighthouse L two ships A and B, which are 600 yards apart, are observed in directions S.W. and 15° East of South respectively. At the same time B is observed from A in a S.E. direction. Draw a plan (scale 1" to 200 yds.), and find by measurement the distance of the lighthouse from each ship.

PARALLELOGRAMS

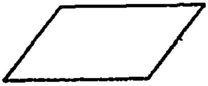
DEFINITIONS

1 A **quadrilateral** is a plane figure bounded by *four* straight lines

The straight line which joins opposite angular points in a quadrilateral is called a **diagonal**

2 A **parallelogram** is a quadrilateral whose opposite sides are *parallel*

[It will be proved hereafter that the opposite sides of a parallelogram are equal, and that its opposite angles are equal]

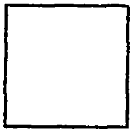


3 A **rectangle** is a parallelogram which has one of its angles a right angle

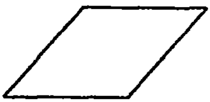
[It will be proved hereafter that *all* the angles of a rectangle are right angles See page 59]

4 A **square** is a rectangle which has two adjacent sides equal

[It will be proved that *all* the sides of a square are equal and all its angles right angles See page 59]



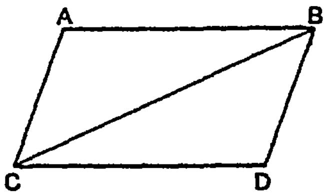
5 A **rhombus** is a quadrilateral which has all its sides equal, but its angles are not right angles



6 A **trapezium** is a quadrilateral which has *one* pair of parallel sides

Theorem 20 [Euclid I 33]

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel



Let AB and CD be equal and parallel straight lines, and let them be joined towards the same parts by the straight lines AC and BD

It is required to prove that AC *and* BD *are equal and parallel*

Join BC

Proof Then because AB and CD are parallel, and BC meets them,

the $\angle$ABC = the alternate $\angle$DCB

Now in the $\triangle$s ABC, DCB,

because $\begin{cases} \text{AB} = \text{DC}, \\ \text{BC is common to both}, \\ \text{and the } \angle\text{ABC} = \text{the } \angle\text{DCB}, \end{cases}$ *Proved*

the triangles are equal in all respects,

so that AC = DB, (i)

and the $\angle$ACB = $\angle$DBC

But these are alternate angles,

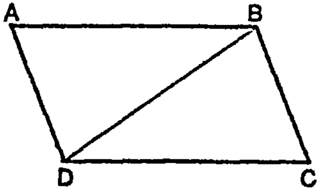
AC and BD are parallel (ii)

That is, AC and BD are both equal and parallel

Q E D

THEOREM 21 [Euclid I 34]

The opposite sides and angles of a parallelogram are equal to one another, and each diagonal bisects the parallelogram



Let ABCD be a parallelogram, of which BD is a diagonal

It is required to prove that

(i) AB = CD, *and* AD = CB,
(ii) *the* $\angle$ BAD = *the* $\angle$ DCB,
(iii) *the* $\angle$ ADC = *the* $\angle$ CBA,
(iv) *the* $\triangle$ ABD = *the* $\triangle$ CDB *in area*

Proof Because AB and DC are parallel, and BD meets them,
the $\angle$ ABD = the alternate $\angle$ CDB

Again, because AD and BC are parallel, and BD meets them,
the $\angle$ ADB = the alternate $\angle$ CBD

Hence in the $\triangle$s ABD, CDB,

because $\begin{cases} \text{the } \angle \text{ ABD} = \text{the } \angle \text{ CDB}, \\ \text{the } \angle \text{ ADB} = \text{the } \angle \text{ CBD}, \\ \text{and BD is common to both} \end{cases}$ *Proved*

the triangles are equal in all respects, *Theor* 17
so that AB = CD, and AD = CB, (i)
and the $\angle$ BAD = the $\angle$ DCB, (ii)
and the $\triangle$ ABD = the $\triangle$ CDB in area (iv)

And because the $\angle$ ADB = the $\angle$ CBD, *Proved*
and the $\angle$ CDB = the $\angle$ ABD,
. the whole $\angle$ ADC = the whole $\angle$ CBA (iii)

Q E D

COROLLARY 1 *If one angle of a parallelogram is a right angle, all its angles are right angles*

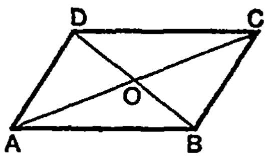
In other words

All *the angles of a rectangle are right angles*

For the sum of two consecutive $\angle^s = 2$ rt. $\angle^s$, (*Theor* 14)
if one of these is a rt angle, the other must be a rt angle
And the opposite angles of the par^m are equal,
all the angles are rt angles

COROLLARY 2 *All the sides of a square are equal, and all its angles are right angles*

COROLLARY 3 *The diagonals of a parallelogram bisect one another*



Let the diagonals AC, BD of the par^m ABCD intersect at O

To prove AO = OC, *and* BO = OD

In the $\triangle^s$ AOB, COD,

because $\begin{cases} \text{the } \angle OAB = \text{the alt } \angle OCD, \\ \text{the } \angle AOB = \text{vert. opp } \angle COD, \\ \quad \text{and } AB = \text{the opp side } CD, \end{cases}$

OA = OC, and OB = OD *Theor* 17

EXERCISES

1 *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram*

2 *If the opposite angles of a quadrilateral are equal, the figure is a parallelogram*

3 *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram*

4 *The diagonals of a rhombus bisect one another at right angles*

5 If the diagonals of a parallelogram are equal, all its angles are right angles

6 In a parallelogram which is not rectangular the diagonals are unequal

EXERCISES ON PARALLELS AND PARALLELOGRAMS

(Symmetry and Superposition.)

1. Shew that by folding a rhombus about one of its diagonals the triangles on opposite sides of the crease may be made to coincide.
That is to say, prove that a rhombus is *symmetrical* about either diagonal.

2. Prove that the diagonals of a square are *axes of symmetry*. Name two other lines about which a square is symmetrical.

3. The diagonals of a rectangle divide the figure into two congruent triangles: is the diagonal, therefore, an axis of symmetry? About what two lines is a rectangle symmetrical?

4. Is there any axis about which an oblique parallelogram is symmetrical? Give reasons for your answer.

5. In a quadrilateral ABCD, AB=AD and CB=CD; but the sides are not all equal. Which of the diagonals (if either) is an axis of symmetry?

6. Prove by the method of superposition that

(i) *Two parallelograms are identically equal if two adjacent sides of one are equal to two adjacent sides of the other, each to each, and one angle of one equal to one angle of the other.*

(ii) *Two rectangles are equal if two adjacent sides of one are equal to two adjacent sides of the other, each to each.*

7. Two quadrilaterals ABCD, EFGH have the sides AB, BC, CD, DA equal respectively to the sides EF, FG, GH, HE, and have also the angle BAD equal to the angle FEH. Shew that the figures may be made to coincide with one another.

(Miscellaneous Theoretical Examples.)

8. Any straight line drawn through the middle point of a diagonal of a parallelogram and terminated by a pair of opposite sides, is bisected at that point.

9. In a parallelogram the perpendiculars drawn from one pair of opposite angles to the diagonal which joins the other pair are equal.

10. If ABCD is a parallelogram, and X, Y respectively the middle points of the sides AD, BC; shew that the figure AYCX is a parallelogram.

11 ABC and DEF are two triangles such that AB, BC are respectively equal to and parallel to DE, EF, shew that AC is equal and parallel to DF

12 ABCD is a quadrilateral in which AB is parallel to DC, and AD equal but not parallel to BC, shew that

(i) the $\angle A +$ the $\angle C = 180° =$ the $\angle B +$ the $\angle D$,

(ii) the diagonal AC = the diagonal BD,

(iii) the quadrilateral is *symmetrical* about the straight line joining the middle points of AB and DC

13 AP, BQ are straight rods of equal length, turning at equal rates (both clockwise) about two fixed pivots A and B respectively If the rods start parallel but pointing in opposite senses, shew that

(i) they will always be parallel,

(ii) the line joining PQ will always pass through a certain fixed point

(Miscellaneous Numerical and Graphical Examples)

14 Calculate the angles of the triangle ABC, having given

$$\text{int. } \angle A = \tfrac{3}{2} \text{ of ext } \angle A, \quad 3B = 4C$$

15 A yacht sailing due East changes her course successively by 63°, by 78°, by 119°, and by 64°, with a view to sailing round an island What further change must be made to set her once more on an Easterly course?

16 If the sum of the interior angles of a rectilineal figure is equal to the sum of the exterior angles, how many sides has it, and why?

17 Draw, using your protractor, any five sided figure ABCDE, in which

$$\angle B = 110°, \quad \angle C = 115°, \quad \angle D = 93°, \quad \angle E = 152°$$

Verify by a construction with ruler and compasses that AE is parallel to BC, and account theoretically for this fact

18 A and B are two fixed points, and two straight lines AP, BQ, unlimited towards P and Q, are pivoted at A and B AP, starting from the direction AB, turns about A clockwise at the uniform rate of $7\tfrac{1}{2}°$ a second and BQ, starting simultaneously from the direction BA, turns about B counter clockwise at the rate of $3\tfrac{3}{4}°$ a second

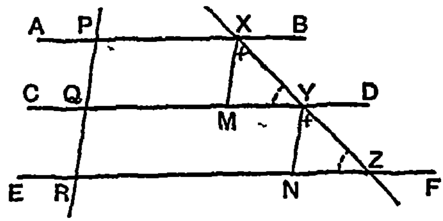
(i) How many seconds will elapse before AP and BQ are parallel?

(ii) Find graphically and by calculation the angle between AP and BQ twelve seconds from the start

(iii) At what rate does this angle decrease?

Theorem 22

If there are three or more parallel straight lines, and the intercepts made by them on any transversal are equal, then the corresponding intercepts on any other transversal are also equal



Let the parallels AB, CD, EF cut off equal intercepts PQ, QR from the transversal PQR, and let XY, YZ be the corresponding intercepts cut off from any other transversal XYZ

It is required to prove that XY = YZ

Through X and Y let XM and YN be drawn parallel to PR

Proof Since CD and EF are parallel, and XZ meets them,
the $\angle$XYM = the corresponding $\angle$YZN
And since XM, YN are parallel, each being parallel to PR,
the $\angle$MXY = the corresponding $\angle$NYZ

Now the figures PM, QN are parallelograms,
∴ XM = the opp side PQ, and YN = the opp side QR,
and since by hypothesis PQ = QR,
XM = YN

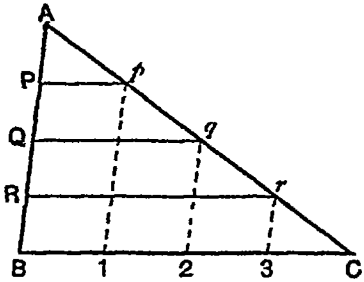
Then in the $\triangle^s$ XMY, YNZ,

because $\begin{cases} \text{the } \angle \text{XYM} = \text{the } \angle \text{YZN}, \\ \text{the } \angle \text{MXY} = \text{the } \angle \text{NYZ}, \\ \quad \text{and XM} = \text{YN}, \end{cases}$

∴ the triangles are identically equal, *Theor* 17
XY = YZ.

Q E D

COROLLARY *In a triangle* ABC, *if a set of lines* Pp, Qq, Rr, ..., *drawn parallel to the base, divide one side* AB *into equal parts, they also divide the other side* AC *into equal parts.*



The lengths of the parallels P*p*, Q*q*, R*r*, ..., may thus be expressed in terms of the base BC

Through *p*, *q*, and *r* let *p*1, *q*2, *r*3 be drawn par[l] to AB

Then by Theorem 22, these par[ls] divide BC into *four* equal parts, of which P*p* evidently contains *one*, Q*q* *two*, and R*r* *three*

In other words,

$$Pp=\frac{1}{4}\ BC,\qquad Qq=\frac{2}{4}\ BC\qquad Rr=\frac{3}{4}\ BC$$

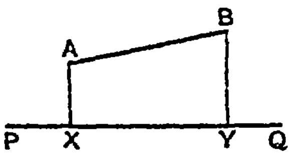
Similarly if the given par[ls] divide AB into n equal parts

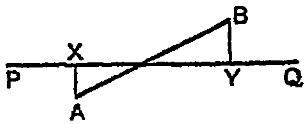
$$Pp=\frac{1}{n}\ BC,\qquad Qq=\frac{2}{n}\ BC,\qquad Rr=\frac{3}{n}\ BC\quad \text{and so on}$$

*** *Problem 7, p 78, should now be worked*

DEFINITION

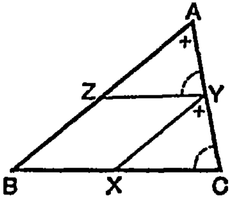
If from the extremities of a straight line AB perpendiculars AX, BY are drawn to a straight line PQ of indefinite length, then XY is said to be the orthogonal projection of AB on PQ.





EXERCISES ON PARALLELS AND PARALLELOGRAMS

1 *The straight line drawn through the middle point of a side of a triangle, parallel to the base, bisects the remaining side.*

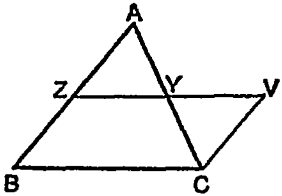


[This is an important particular case of Theorem 22

In the $\triangle$ABC, if Z is the middle point of AB, and ZY is drawn par[1] to BC, we have to prove that AY = YC

Draw YX par[1] to AB, and then prove the $\triangle$s ZAY, XYC congruent]

2 *The straight line which joins the middle points of two sides of a triangle is parallel to the third side*



[In the $\triangle$ABC, if Z, Y are the middle points of AB, AC, we have to prove ZY par[1] to BC

Produce ZY to V, making YV equal to ZY, and join CV Prove the $\triangle$s AYZ, CYV congruent, the rest follows at once]

3 *The straight line which joins the middle points of two sides of a triangle is equal to half the third side*

4 *Shew that the three straight lines which join the middle points of the sides of a triangle, divide it into four triangles which are identically equal*

5 *Any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other sides of the triangle.*

6 ABCD is a parallelogram, and X, Y are the middle points of the opposite sides AD, BC shew that BX and DY trisect the diagonal AC

7 *If the middle points of adjacent sides of any quadrilateral are joined, the figure thus formed is a parallelogram*

8 Shew that the straight lines which join the middle points of opposite sides of a quadrilateral, bisect one another

9 From two points A and B and from O the mid point between them perpendiculars AP, BQ, OX are drawn to a straight line CD If AP BQ measure respectively 4 2 cm and 5 8 cm, *deduce* the length of OX, and verify your result by measurement

Shew that $OX = \frac{1}{2}(AP + BQ)$ or $\frac{1}{2}(AP - BQ)$, according as A and B are on the *same* side or on *opposite* sides of CD

10 When three parallels cut off equal intercepts from two transversals shew that of the three parallel lengths between the two transversals the middle one is the **Arithmetic Mean** of the other two

11 *The parallel sides of a trapezium are* a *centimetres and* b *centimetres in length. Prove that the line joining the middle points of the oblique sides is parallel to the parallel sides, and that its length is* $\frac{1}{2}(a+b)$ *centimetres*

12. OX and OY are two straight lines, and along OX five points 1, 2 3, 4, 5 are marked at equal distances Through these points parallels are drawn in any direction to meet OY Measure the lengths of these parallels take their average, and compare it with the length of the third parallel Prove *geometrically* that the 3rd parallel is the mean of all five

State the corresponding theorem for any odd number $(2n-1)$ of parallels so drawn

13 From the angular points of a parallelogram perpendiculars are drawn to any straight line which is outside the parallelogram shew that the sum of the perpendiculars drawn from one pair of opposite angles is equal to the sum of those drawn from the other pair

[Draw the diagonals and from their point of intersection suppose a perpendicular drawn to the given straight line.]

14 The sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the equal sides is equal to the perpendicular drawn from either extremity of the base to the opposite side

[It follows that the sum of the distances of *any* point in the base of an isosceles triangle from the equal sides is constant, that is, the same whatever point in the base is taken]

How would this property be modified if the given point were taken in the base *produced*?

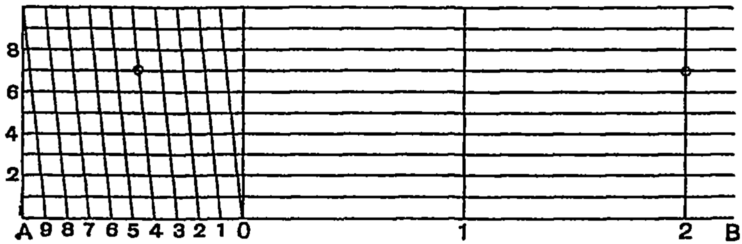
15 The sum of the perpendiculars drawn from any point within an equilateral triangle to the three sides is equal to the perpendicular drawn from any one of the angular points to the opposite side, and is therefore constant

16 Equal and parallel lines have equal projections on any other straight line

DIAGONAL SCALES

Diagonal scales form an important application of Theorem 22 We shall illustrate their construction and use by describing a *Decimal Diagonal Scale to shew Inches, Tenths, and Hundredths*

A straight line AB is divided (from A) into inches, and the points of division marked 0, 1, 2 The primary division 0A is subdivided into *tenths*, these secondary divisions being numbered (from 0) 1, 2, 3, 9 We may now read on AB *inches* and *tenths* of an inch



In order to read *hundredths*, ten lines are taken at any equal intervals parallel to AB, and perpendiculars are drawn through 0, 1, 2,

The primary (or inch) division corresponding to 0A on the tenth parallel is now subdivided into *ten* equal parts, and diagonal lines are drawn, as in the diagram,

joining 0 to the *first* point of subdivision on the 10th parallel
„ 1 to the *second* „ „ „ „ ,
„ 2 to the *third* „ „ „ „ ,
and so on

The scale is now complete, and its use is shewn in the following example

Example *To take from the scale a length of* 2 47 *inches*

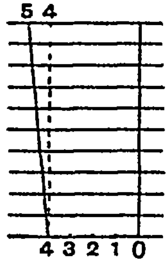
(i) Place one point of the dividers at 2 in AB, and extend them till the other point reaches 4 in the subdivided inch 0A We have now 2 4 inches in the dividers

(ii) To get the remaining 7 *hundredths*, move the right-hand point up the perpendicular through 2 till it reaches the 7th parallel Then extend the dividers till the left point reaches the diagonal 4 also on the 7th parallel We have now 2 47 inches in the dividers.

REASON FOR THE ABOVE PROCESS

The first step needs no explanation. The reason of the second is found in the Corollary of Theorem 22.

Joining the point 4 to the corresponding point on the tenth parallel, we have a triangle 4,4,5, of which one side 4,4 is divided into ten equal parts by a set of lines parallel to the base 4,5.



Therefore the lengths of the parallels between 4,4, and the diagonal 4,5 are $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \ldots$ of the base, which is 1 inch.

Hence these lengths are respectively
.01, .02, .03, ... of 1 inch.

Similarly, by means of this scale, the length of a given straight line may be measured to the nearest hundredth of an inch.

Again, if one inch-division on the scale is taken to represent 10 feet, then 2.47 inches on the scale will represent 24.7 feet. And if one inch-division on the scale represents 100 links, then 2.47 inches will represent 247 links. Thus a diagonal scale is of service in preparing plans of enclosures, buildings, or field works, where it is necessary that every dimension of the actual object must be represented by a line of proportional length on the plan.

NOTE.

The subdivision of a diagonal scale need not be *decimal*.

For instance we might construct a diagonal scale to read centimetres, millimetres, and *quarters* of a millimetre, in which case we should take *four* parallels to the line AB.

[For Exercises on Linear Measurements see the following page.]

EXERCISES ON LINEAR MEASUREMENTS

1 Draw straight lines whose lengths are 1 25 inches, 2·72 inches, 3 08 inches

2 Draw a line 2 68 inches long, and measure its length in centimetres and the nearest millimetre

3 Draw a line 5 7 cm in length, and measure it in inches (to the nearest hundredth) Check your result by calculation, given that 1 cm = 0 3937 inch

4 Find by measurement the equivalent of 3 15 inches in centimetres and millimetres Hence calculate (correct to two decimal places) the value of 1 cm in inches

5 Draw lines 2·9 cm and 6·2 cm in length, and measure them in inches Use each equivalent to find the value of 1 inch in centimetres and millimetres, and take the average of your results

6 A distance of 100 miles is represented on a map by 1 inch Draw lines to represent distances of 336 miles and 408 miles

7 If 1 inch on a map represents 1 kilometre, draw lines to represent 850 metres, 2980 metres, and 1010 metres

8 A plan is drawn to the scale of 1 inch to 100 links Measure in centimetres and millimetres a line representing 417 links

9 Find to the nearest hundredth of an inch the length of a line which will represent 42 500 kilometres in a map drawn to the scale of 1 centimetre to 5 kilometres

10 The distance from London to Oxford (in a direct line) is 55 miles. If this distance is represented on a map by 2 75 inches, to what scale is the map drawn? That is, how many miles will be represented by 1 inch? How many kilometres by 1 centimetre?
[1 cm = 0 3937 inch, 1 km. = $\frac{5}{8}$ mile, nearly]

11 On a map of France drawn to the scale 1 inch to 35 miles, the distance from Paris to Calais is represented by 4 2 inches Find the distance accurately in miles, and approximately in kilometres, and express the scale in metric measure [1 km = $\frac{5}{8}$ mile, nearly]

12 The distance from Exeter to Plymouth is 37½ miles, and appears on a certain map to be 2½″, and the distance from Lincoln to York is 88 km , and appears on another map to be 7 cm Compare the scales of these maps in miles to the inch

13 Draw a diagonal scale, 2 centimetres to represent 1 yard, shewing yards, feet, and inches.

Raj Bahadur Mathur.

PRACTICAL GEOMETRY

PROBLEMS

The following problems are to be solved with ruler and compasses only No step requires the actual measurement of any line or angle, that is to say, the *constructions* are to be made without using either a graduated scale of length, or a protractor

The problems are not merely to be studied as propositions, but the construction in every case is to be actually performed by the learner, great care being given to accuracy of drawing

Each problem is followed by a *theoretical* proof, but the *results* of the work should always be verified by measurement, as a test of correct drawing Accurate measurement is also required in applications of the problems

In the diagrams of the problems lines which are inserted only for purposes of *proof* are dotted, to distinguish them from lines necessary to the construction

For practical applications of the problems the student should be provided with the following instruments

1 A flat ruler, one edge being graduated in centimetres and millimetres, and the other in inches and tenths

2 Two set squares, one with angles of 45°, and the other with angles of 60° and 30°

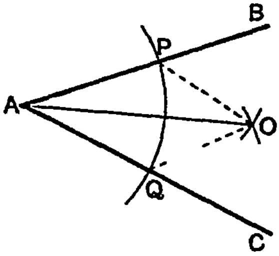
3 A pair of pencil compasses

4 A pair of dividers, preferably with screw adjustment.

5 A semi circular protractor

PROBLEM 1

To bisect a given angle



Let BAC be the given angle to be bisected

Construction With centre A, and any radius, draw an arc of a circle cutting AB, AC at P and Q.

With centres P and Q, and radius PQ, draw two arcs cutting at O

Join AO

Then the $\angle$ BAC is bisected by AO

Proof Join PO, QO

In the $\triangle^s$ APO, AQO,

because $\begin{cases} \text{AP} = \text{AQ, being radii of a circle,} \\ \text{PO} = \text{QO, ,, ,, equal circles,} \\ \text{and AO is common,} \end{cases}$

the triangles are equal in all respects, *Theor* 7

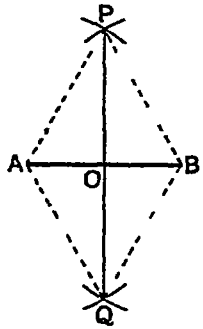
so that the $\angle$ PAO = the $\angle$ QAO,

that is, the $\angle$ BAC is bisected by AO

NOTE PQ has been taken as the radius of the arcs drawn from the centres P and Q, and the intersection of these arcs determines the point O *Any* radius, however, may be used instead of PQ, provided that it is great enough to secure the intersection of the arcs

PROBLEM 2

To bisect a given straight line



Let AB be the line to be bisected

Construction. With centre A, and radius AB, draw two arcs, one on each side of AB

With centre B, and radius BA, draw two arcs, one on each side of AB, cutting the first arcs at P and Q.

Join PQ, cutting AB at O

Then AB is bisected at O

Proof. Join AP, AQ, BP, BQ

In the $\triangle^s$ APQ, BPQ,

because AP = BP, being radii of equal circles,
AQ = BQ, for the same reason,
and PQ is common,

the $\angle APQ$ = the $\angle BPQ$. *Theor* 7

Again in the $\triangle^s$ APO, BPO,

because AP = BP,
PO is common,
and the $\angle APO$ = the $\angle BPO$,

AO = OB, *Theor* 4

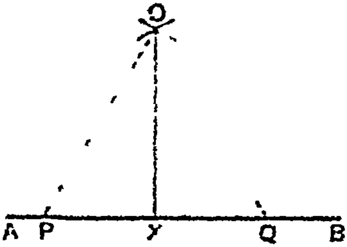
that is, AB is bisected at O

NOTES (i) AB was taken as the radius of the arcs drawn from the centres A and B, but *any* radius may be used provided that it is great enough to secure the intersection of the arcs which determine the points P and Q.

(ii) From the congruence of the $\triangle^s$ APO, BPO it follows that the $\angle AOP$ = the $\angle BOP$ As these are adjacent angles, it follows that PQ *bisects* AB *at right angles*

Problem 3

To draw a straight line perpendicular to a given straight line at a given point in it



Let AB be the straight line, and X the point in it at which a perpendicular is to be drawn

Construction With centre X cut off from AB any two equal parts XP, XQ.

With centres P and Q and radius PQ draw two arcs cutting at O

Join XO

Then XO is perp to AB

Proof Join OP, OQ

In the △s OXP, OXQ

because { XP = XQ, by construction, OX is common, and PO = QO, being radii of equal circles

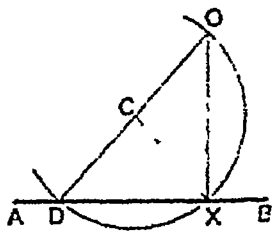
the ∠OXP = the ∠OXQ *Theor* 7

And these being adjacent angles, each is a right angle,

that is, XO is perp to AB

Obs If the point X is near one end of AB one or other of the alternative constructions on the next page should be used

PROBLEM 3 SECOND METHOD



Construction. Take any point C outside AB

With centre C, and radius CX, draw a circle cutting AB at D

Join DC, and produce it to meet the circumference of the circle at O

Join XO

Then XO is perp to AB

Proof. Join CX

Because CO = CX, the ∠CXO = the ∠COX,

and because CD = CX, the ∠CXD = the ∠CDX

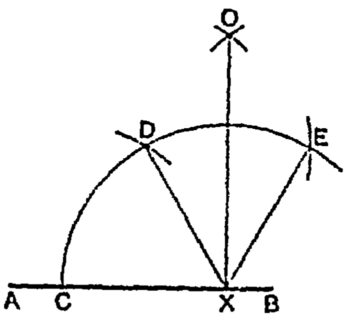
∴ the whole ∠DXO = the ∠XOD + the ∠XDO

$= \frac{1}{2}$ of 180°

= 90°

XO is perp to AB

PROBLEM 3 THIRD METHOD



Construction With centre X and any radius, draw the arc CDE, cutting AB at C

With centre C, and with the same radius, draw an arc, cutting the first arc at D

With centre D, and with the same radius, draw an arc cutting the first arc at E

Bisect the ∠DXE by XO *Prob* 1

Then XO is perp to AB

Proof Each of the ∠s CXD, DXE may be proved to be 60°

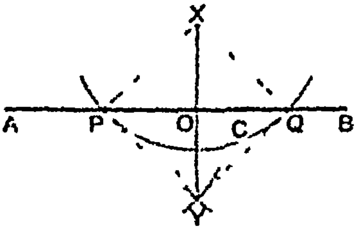
and the ∠DXO is half of the ∠DXE,

the ∠CXO is 90°

That is, XO is perp to AB

Problem 4

To draw a straight line perpendicular to a given straight line from a given external point



Let X be the given external point from which a perpendicular is to be drawn to AB

Construction Take any point C on the side of AB remote from X

With centre X, and radius XC, draw an arc to cut AB at P and Q

With centres P and Q and radius PX draw arcs cutting at Y, on the side of AB opposite to X

Join XY cutting AB at O

Then XO is perp to AB

Proof Join PX QX PY, QY,

In the △s PXY, QXY,

because $\begin{cases} PX = QX, \text{ being radii of a circle,} \\ PY = QY, \text{ for the same reason,} \\ \text{and } XY \text{ is common} \end{cases}$

the ∠PXY = the ∠QXY *Theor* 7

Again, in the △s PXO, QXO,

because $\begin{cases} PX = QX, \\ XO \text{ is common,} \\ \text{and the } \angle PXO = \text{the } \angle QXO \end{cases}$

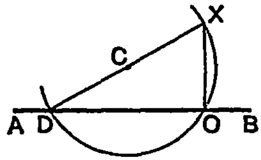
the ∠XOP = the ∠XOQ *Theor* 4

And these being adjacent angles, each is a right angle,

that is, XO is perp to AB

Obs When the point X is nearly opposite one end of AB, one or other of the alternative constructions given below should be used.

PROBLEM 4 SECOND METHOD



Construction Take any point D in AB Join DX, and bisect it at C

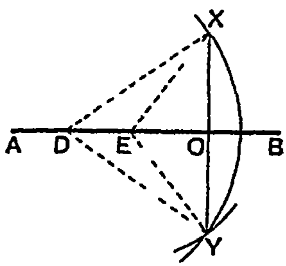
With centre C, and radius CX, draw a circle cutting AB at D and O

Join XO

Then XO is perp to AB

For, as in Problem 3, Second Method, the $\angle$ XOD is a right angle

PROBLEM 4 THIRD METHOD



Construction Take any two points D and E in AB

With centre D, and radius DX, draw an arc of a circle, on the side of AB opposite to X

With centre E, and radius EX, draw another arc cutting the former at Y

Join XY, cutting AB at O

Then XO is perp to AB

(i) Prove the $\triangle^s$ XDE, YDE equal in all respects by Theorem 7,

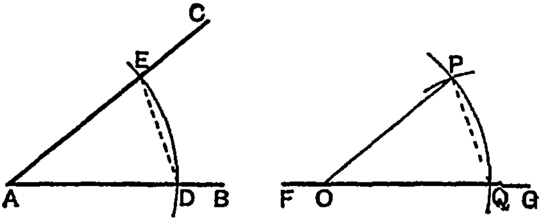
so that the $\angle$ XDE = the $\angle$ YDE

(ii) Hence prove the $\triangle^s$ XDO, YDO equal in all respects by Theorem 4, so that the adjacent $\angle^s$ DOX, DOY are equal

That is, XO is perp to AB

PROBLEM 5

At a given point in a given straight line to make an angle equal to a given angle



Let BAC be the given angle, and FG the given straight line, and let O be the point at which an angle is to be made equal to the $\angle$ BAC

Construction With centre A, and with any radius, draw an arc cutting AB and AC at D and E

With centre O, and with the same radius, draw an arc cutting FG at Q.

With centre Q, and with radius DE, draw an arc cutting the former arc at P

Join OP

Then POQ is the required angle.

Proof Join ED, PQ.

In the $\triangle$ˢ POQ, EAD,

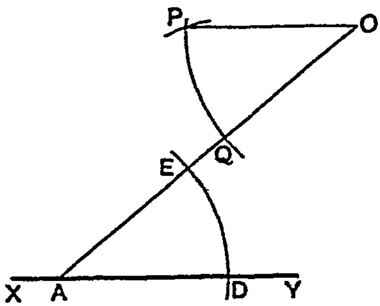
because $\begin{cases} \text{OP} = \text{AE, being radii of equal circles,} \\ \text{OQ} = \text{AD, for the same reason,} \\ \text{PQ} = \text{ED, by construction,} \end{cases}$

the triangles are equal in all respects,

so that the $\angle$ POQ = the $\angle$ EAD *Theor* 7

PROBLEM 6

Through a given point to draw a straight line parallel to a given straight line.



Let XY be the given straight line, and O the given point, through which a straight line is to be drawn par[l] to XY

Construction. In XY take any point A, and join OA

Using the construction of Problem 5, at the point O in the line AO make the $\angle$AOP equal to the $\angle$OAY and alternate to it

Then OP is parallel to XY

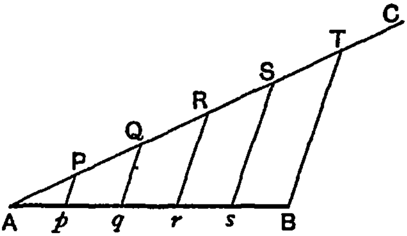
Proof Because AO, meeting the straight lines OP, XY, makes the alternate $\angle$[s] POA, OAY equal,

OP is par[l] to XY

⁂ *The constructions of Problems 3, 4, and 6 are not usually followed in practical applications Parallels and perpendiculars may be more quickly drawn by the aid of set squares* (*See* LESSONS IN EXPERIMENTAL GEOMETRY, *pp* 36, 42)

PROBLEM 7

To divide a given straight line into any number of equal parts



Let AB be the given straight line, and suppose it is required to divide it into *five* equal parts

Construction. From A draw AC, a straight line of unlimited length, making any angle with AB

From AC mark off *five* equal parts of *any* length, AP, PQ, QR, RS, ST

Join TB, and through P, Q, R, S draw par[ls] to TB, meeting AB in *p*, *q*, *r*, *s*

Then since the par[ls] P*p*, Q*q*, R*r*, S*s*, TB cut off five equal parts from AT, they also cut off five equal parts from AB (Theorem 22)

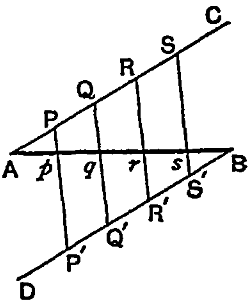
SECOND METHOD

From A draw AC at any angle with AB, and on it mark off *four* equal parts AP, PQ, QR, RS, of any length

From B draw BD par[l] to AC, and on it mark off BS′, S′R′, R′Q′, Q′P′, each equal to the parts marked on AC

Join PP′, QQ′, RR′, SS′ meeting AB in *p*, *q*, *r*, *s* Then AB is divided into five equal parts at these points

[Prove by Theorems 20 and 22]



EXERCISES ON LINES AND ANGLES

(*Graphical Exercises.*)

1. Construct (with ruler and compasses only) an angle of 60°.
By repeated bisection divide this angle into *four* equal parts.

2. By means of Exercise 1, *trisect a right angle*, that is, divide it into *three* equal parts.

Bisect each part, and hence show how to trisect an angle of 45°.
[No construction is known for exactly trisecting *any* angle.]

3. Draw a line 6·7 cm. long, and divide it into *five* equal parts. Measure one of the parts in inches (to the nearest hundredth), and verify your work by calculation. [1 cm. = 0·3937 inch.]

4. From a straight line 3·72″ long, cut off *one seventh*. Measure the part in centimetres and the nearest millimetre, and verify your work by calculation.

5. At a point X in a straight line AB draw XP perpendicular to AB, making XP 1·8″ in length. From P draw an oblique PQ, 3·0″ long, to meet AB in Q. Measure XQ.

(*Problems. State your construction, and give a theoretical proof.*)

6. In a straight line XY find a point which is equidistant from two given points A and B.
When is this impossible?

7. In a straight line XY find a point which is equidistant from two intersecting lines AB, AC.
When is this impossible?

8. From a given point P draw a straight line PQ, making with a given straight line AB an angle of given magnitude.

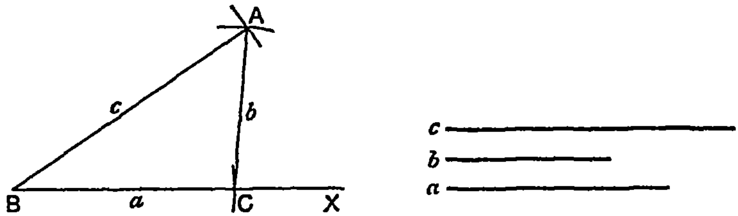
9. From two given points P and Q on the same side of a straight line AB, draw two lines which meet in AB and make equal angles with it.
[*Construction.* From P draw PH perp. to AB, and produce PH to P′, making HP′ equal to PH. Join P′Q cutting AB at K. Join PK. Prove that PK, QK are the required lines.]

10. Through a given point P draw a straight line such that the perpendiculars drawn to it from two points A and B may be equal.
Is this always possible?

The Construction of Triangles

Problem 8

To draw a triangle having given the lengths of the three sides



Let a, b, c be the lengths to which the sides of the required triangle are to be equal

Construction Draw any straight line BX, and cut off from it a part BC equal to a

With centre B, and radius c, draw an arc of a circle

With centre C, and radius b, draw a second arc cutting the first at A

Join AB, AC

Then ABC is the required triangle, for by construction the sides BC, CA, AB are equal to a, b, c respectively

Obs The three data, a, b, c may be understood in two ways either as three actual lines to which the sides of the triangle are to be equal, or as three *numbers* expressing the lengths of those lines in terms of inches, centimetres, or some other linear unit

Notes (i) In order that the construction may be possible it is necessary that any two of the given sides should be together greater than the third side (Theorem 11), for otherwise the arcs drawn from the centres B and C would not cut

(ii) The arcs which cut at A would, if continued, cut again on the other side of BC Thus the construction gives two triangles on opposite sides of a common base

It has been seen (Page 50) that to prove two triangles identically equal, *three* parts of one must be given equal to the corresponding parts of the other (though *any* three parts do not necessarily serve the purpose) This amounts to saying that *to determine the shape and size of a triangle we must know three of its parts* or, in other words,

To construct a triangle three independent data are required

For example, we may construct a triangle

(i) When *two sides* (b, c) and the *included angle* (A) are given.

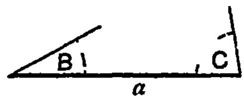
The method of construction in this case is obvious.

(ii) When *two angles* (A, B) and one side (a) are given

Here, since A and B are given, we at once know C,

for $A+B+C=180°$

Hence we have only to draw the base equal to a, and at its ends make angles equal to B and C, for we know that the remaining angle must necessarily be equal to A



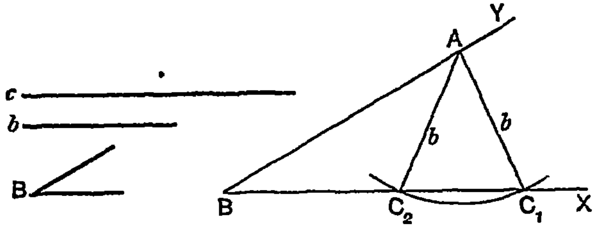
(iii) If the *three angles* A, B, C are given (and no side), the problem is indeterminate, that is, the number of solutions is unlimited

For if at the ends of *any* base we make angles equal to B and C, the third angle is equal to A

This construction is indeterminate, because the three data are not *independent*, the third following necessarily from the other two

Problem 9

To construct a triangle having given two sides and an angle opposite to one of them



Let b, c be the given sides and B the given angle

Construction Take any straight line BX, and at B make the $\angle$XBY equal to the given $\angle$B

From BY cut off BA equal to c

With centre A, and radius b, draw an arc of a circle

If this arc cuts BX in two points C_1 and C_2, both on the same side of B, both the $\triangle^s$ ABC_1, ABC_2 satisfy the given conditions

This double solution is known as the **Ambiguous Case**, and will occur when b is less than c but greater than the perp from A on BX

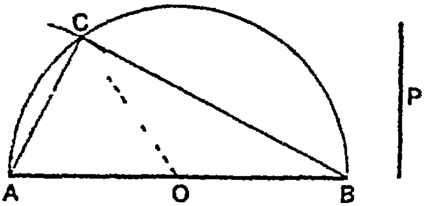
EXERCISE.

Draw figures to illustrate the nature and number of solutions in the following cases

(i) When b is greater than c
(ii) When b is equal to c
(iii) When b is equal to the perpendicular from A on BX
(iv) When b is less than this perpendicular

Problem 10

To construct a right angled triangle having given the hypotenuse and one side.



Let AB be the hypotenuse and P the given side.

Construction. Bisect AB at O, and with centre O, and radius OA, draw a semicircle.

With centre A, and radius P, draw an arc to cut the semicircle at C.

Join AC, BC.

Then ABC is the required triangle.

Proof. Join OC.

Because OA = OC,
the ∠OCA = the ∠OAC.

And because OB = OC,
the ∠OCB = the ∠OBC.

∴ the whole ∠ACB = the ∠OAC + the ∠OBC
$= \frac{1}{2}$ of 180° *Theor.* 16.
= 90°.

ON THE CONSTRUCTION OF TRIANGLES

(Graphical Exercises.)

1 Draw a triangle whose sides are 7 5 cm, 6·2 cm, and 5 3 cm Draw and measure the perpendiculars dropped on these sides from the opposite vertices

[N B The perpendiculars, if correctly drawn, will meet at a point, as will be seen later See page 207]

2 Draw a triangle ABC, having given $a=3\ 00''$, $b=2\ 50''$, $c=2\ 75''$ Bisect the angle A by a line which meets the base at X Measure BX and XC (to the nearest hundredth of an inch), and hence calculate the value of $\frac{BX}{CX}$ to two places of decimals Compare your result with the value of $\frac{c}{b}$

3 Two sides of a triangular field are 315 yards and 260 yards, and the included angle is known to be 39° Draw a plan (1 inch to 100 yards) and find by measurement the length of the remaining side of the field

4 ABC is a triangular plot of ground, of which the base BC is 75 metres, and the angles at B and C are 47° and 68° respectively Draw a plan (scale 1 cm. to 10 metres) Write down without measurement the size of the angle A, and by measuring the plan, obtain the approximate lengths of the other sides of the field, also the perpendicular drawn from A to BC

5 A yacht on leaving harbour steers N E sailing 9 knots an hour After 20 minutes she goes about, steering N W for 35 minutes and making the same average speed as before How far is she now from the harbour, and what course (approximately) must she set for the run home? Obtain your results from a chart of the whole course, scale 2 cm to 1 knot

6 Draw a right-angled triangle, given that the hypotenuse $c=10\ 6$ cm and one side $a=5\ 6$ cm Measure the third side b, and find the value of $\sqrt{c^2-a^2}$ Compare the two results

7 Construct a triangle, having given the following parts B=34°, $b=5\ 5$ cm, $c=8\ 5$ cm Shew that there are two solutions Measure the two values of a, and also of C, and shew that the latter are supplementary

8 In a triangle ABC, the angle A=50°, and $b=6\ 5$ cm. Illustrate by figures the cases which arise in constructing the triangle, when (i) $a=7$ cm (ii) $a=6$ cm (iii) $a=5$ cm. (iv) $a=4$ cm.

9 Two straight roads, which cross at right angles at A, are carried over a straight canal by bridges at B and C The distance between the bridges is 461 yards, and the distance from the crossing A to the bridge B is 261 yards Draw a plan, and by measurement of it ascertain the distance from A to C

(*Problems State your construction, and give a theoretical proof*)

10 Draw an isosceles triangle on a base of 4 cm, and having an altitude of 6 2 cm. *Prove* the two sides equal, and measure them to the nearest millimetre

11 Draw an isosceles triangle having its vertical angle equal to a given angle, and the perpendicular from the vertex on the base equal to a given straight line

Hence draw an equilateral triangle in which the perpendicular from one vertex on the opposite side is 6 cm Measure the length of a side to the nearest millimetre

12 Construct a triangle ABC in which the perpendicular from A on BC is 5 0 cm, and the sides AB, AC are 5 8 cm and 9·0 cm respectively Measure BC

13 Construct a triangle ABC having the angles at B and C equal to two given angles L and M, and the perpendicular from A on BC equal to a given line P

14 Construct a triangle ABC (without protractor) having given two angles B and C and the side b

15 On a given base construct an isosceles triangle having its vertical angle equal to a given angle L

16 Construct a right angled triangle, having given the length of the hypotenuse c, and the sum of the remaining sides a and b

If $c=5\ 3$ cm, and $a+b=7\ 3$ cm, find a and b graphically, and calculate the value of $\sqrt{a^2+b^2}$

17 Construct a triangle having given the perimeter and the angles at the base For example, $a+b+c=12$ cm, $B=70°$, $C=80°$

18 Construct a triangle ABC from the following data

$$a=6\ 5 \text{ cm}, \quad b+c=10 \text{ cm}, \quad \text{and } B=60°$$

Measure the lengths of b and c.

19 Construct a triangle ABC from the following data

$$a=7 \text{ cm}, \quad c-b=1 \text{ cm}, \quad \text{and } B=55°$$

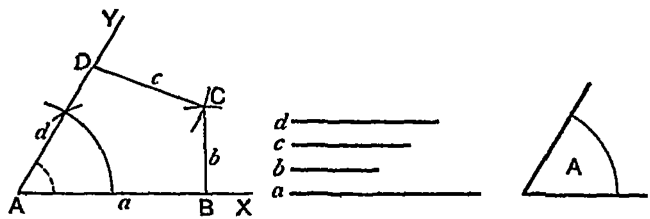
Measure the lengths of b and c

THE CONSTRUCTION OF QUADRILATERALS

It has been shewn that the shape and size of a triangle are completely determined when the lengths of its three sides are given A quadrilateral, however, is not completely determined by the lengths of its four sides From what follows it will appear that *five* independent data are required to construct a quadrilateral

Problem 11

To construct a quadrilateral, given the lengths of the four sides and one angle



Let a, b, c, d be the given lengths of the sides, and A the angle between the sides equal to a and d

Construction. Take any straight line AX, and cut off from it AB equal to a

Make the $\angle$ BAY equal to the $\angle$ A.

From AY cut off AD equal to d

With centre D, and radius c, draw an arc of a circle

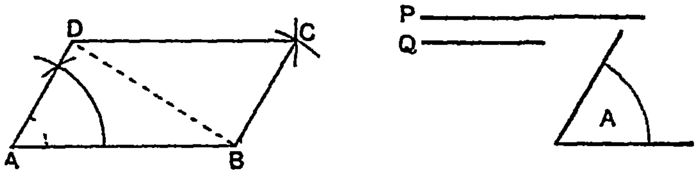
With centre B and radius b, draw another arc to cut the former at C

Join DC, BC

Then ABCD is the required quadrilateral, for by construction the sides are equal to a, b, c, d, and the $\angle$ DAB is equal to the given angle

PROBLEM 12

To construct a parallelogram having given two adjacent sides and the included angle



Let P and Q be the two given sides, and A the given angle

Construction 1 (*With ruler and compasses*) Take a line AB equal to P, and at A make the ∠ BAD equal to the ∠ A, and make AD equal to Q.

With centre D, and radius P, draw an arc of a circle

With centre B, and radius Q, draw another arc to cut the former at C

Then ABCD is the required par^m

Proof. Join DB.

In the △^s DCB, BAD,

because { DC = BA, CB = AD, and DB is common,

the ∠ CDB = the ∠ ABD, *Theor* 7

and these are alternate angles,

DC is par^l to AB

Also DC = AB,

∴ DA and BC are also equal and parallel *Theor* 20

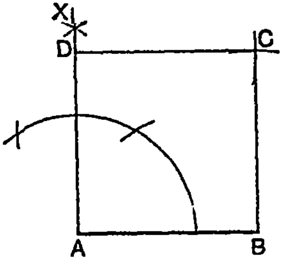
ABCD is a par^m

Construction 2 (*With set squares*) Draw AB and AD as before, then with set squares through D draw DC par^l to AB, and through B draw BC par^l to AD

By construction, ABCD is a par^m having the required parts

PROBLEM 13

To construct a square on a given side



Let AB be the given side

Construction 1 (*With ruler and compasses*) At A draw AX perp to AB, and cut off from it AD equal to AB

With B and D as centres, and with radius AB, draw two arcs cutting at C

Join BC, DC

Then ABCD is the required square

Proof. As in Problem 12, ABCD may be shewn to be a par^m^
And since the ∠BAD is a right angle, the figure is a rectangle
Also, by construction all its sides are equal

ABCD is a square

Construction 2 (*With set squares*) At A draw AX perp to AB, and cut off from it AD equal to AB

Through D draw DC par^l^ to AB, and through B draw BC par^l^ to AD meeting DC in C

Then, by construction, ABCD is a rectangle [Def 3, page 56]

Also it has the two adjacent sides AB, AD equal

it is a square.

EXERCISES

ON THE CONSTRUCTION OF QUADRILATERALS

1 Draw a rhombus each of whose sides is equal to a given straight line PQ, which is also to be one diagonal of the figure

Ascertain (without measurement) the number of degrees in each angle, giving a reason for your answer

2 Draw a square on a side of 2 5 inches Prove theoretically that its diagonals are equal, and by measuring the diagonals to the nearest hundredth of an inch test the correctness of your drawing

3 Construct a square on a diagonal of 3·0″, and measure the lengths of each side Obtain the average of your results

4 Draw a parallelogram ABCD, having given that one side AB=5 5 cm, and the diagonals AC, BD are 8 cm, and 6 cm respectively Measure AD

5 The diagonals of a certain quadrilateral are equal, (each 6 0 cm.), and they bisect one another at an angle of 60° Shew that *five* independent data are here given

Construct the quadrilateral Name its species, and give a *formal* proof of your answer Measure the perimeter If the angle between the diagonals were increased to 90°, by how much per cent would the perimeter be increased?

6 In a quadrilateral ABCD,

AB=5·6 cm., BC=2 5 cm, CD=4·0 cm, and DA=3 3 cm

Shew that the shape of the quadrilateral is not settled by these data

Draw the quadrilateral when (i) A=30° (ii) A=60° Why does the construction fail when A=100°?

Determine graphically the least value of A for which the construction fails

7 Shew how to construct a quadrilateral, having given the lengths of the four sides and of one diagonal What conditions must hold among the data in order that the problem may be possible?

Illustrate your method by constructing a quadrilateral ABCD, when

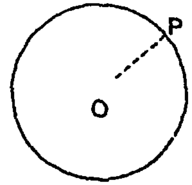
(i) AB=3 0″, BC=1 7″, CD=2 5″, DA=2 8″, and the diagonal BD=2·6″ Measure AC

(ii) AB=3 6 cm, BC=7 7 cm, CD=6 8 cm, DA=5 1 cm, and the diagonal AC=8 5 cm Measure the angles at B and D

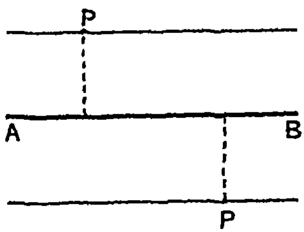
LOCI

DEFINITION The locus of a point is the path traced out by it when it moves in accordance with some given law

Example 1 Suppose the point P to move so that its distance from a fixed point O is constant (say 1 centimetre)
Then the locus of P is evidently the circumference of a circle whose centre is O and radius 1 cm



Example 2 Suppose the point P moves at a constant distance (say 1 cm) from a fixed straight line AB
Then the locus of P is one or other of two straight lines parallel to AB, on either side, and at a distance of 1 cm from it

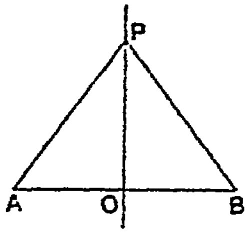


Thus the locus of a point, moving under some given condition, consists of the line or lines to which the point is thereby restricted, provided that the condition is satisfied by every point on such line or lines, and by no other

When we find a series of points which satisfy the given law, and through which therefore the moving point must pass, we are said to plot the locus of the point

PROBLEM 14

To find the locus of a point P which moves so that its distances from two fixed points A and B are always equal to one another



Here the point P moves through all positions in which PA = PB,
one position of the moving point is at O the middle point of AB

Suppose P to be *any* other position of the moving point, that is, let PA = PB

Join OP

Then in the $\triangle$s POA, POB,

because $\begin{cases} \text{PO is common,} \\ \text{OA = OB,} \\ \text{and PA = PB, by hypothesis} \end{cases}$

the $\angle$ POA = the $\angle$ POB *Theor* 7

Hence PO is perpendicular to AB

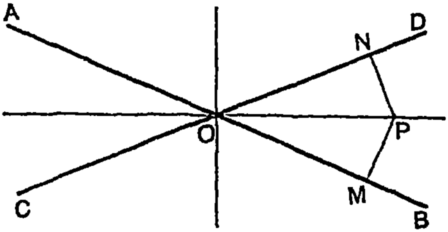
That is *every point* P *which is equidistant from* A *and* B *lies on the straight line bisecting* AB *at right angles*

Likewise it may be proved that every point on the perpendicular through O is equidistant from A and B

This line is therefore the required locus

Problem 15

To find the locus of a point P *which moves so that its perpendicular distances from two given straight lines* AB, CD *are equal to one another*



Let P be any point such that the perp PM = the perp PN

Join P to O, the intersection of AB, CD

Then in the △ˢ PMO, PNO,

because $\begin{cases}\text{the } \angle^s \text{ PMO, PNO are } \textit{right angles,} \\ \text{the hypotenuse OP is common,} \\ \text{and one side PM = one side PN,}\end{cases}$

the triangles are equal in all respects, *Theor* 18

so that the ∠ POM = the ∠ PON

Hence, if P lies within the ∠ BOD, it must be on the bisector of that angle,

and, if P is within the ∠ AOD, it must be on the bisector of that angle

It follows that *the required locus is the pair of lines which bisect the angles between* AB *and* CD

INTERSECTION OF LOCI

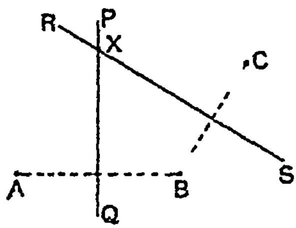
The method of Loci may be used to find the position of a point which is subject to two conditions. For corresponding to each condition there will be a locus on which the required point must lie. Hence all points which are common to these two loci, that is, all the points of intersection of the loci, will satisfy *both* the given conditions.

EXAMPLE 1. *To find a point equidistant from three given points A, B, C, which are not in the same straight line.*

(i) The locus of points equidistant from A and B is the straight line PQ, which bisects AB at right angles.

(ii) Similarly, the locus of points equidistant from B and C is the straight line RS which bisects BC at right angles.

Hence the point common to PQ and RS must satisfy both conditions: that is to say, X the point of intersection of PQ and RS will be equidistant from A, B, and C.



EXAMPLE 2. *To construct a triangle, having given the base, the altitude, and the length of the median which bisects the base.*

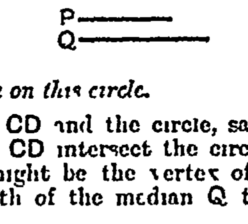
Let AB be the given base, and P and Q the lengths of the altitude and median respectively.

Then the triangle is known if its *vertex* is known.

(i) Draw a straight line CD parallel to AB, and at a distance from it equal to P: *then the required vertex must lie on* CD.

(ii) Again, from O the middle point of AB as centre, with radius equal to Q, describe a circle:

then the required vertex must lie on this circle.



Hence any points which are common to CD and the circle, satisfy both the given conditions: that is to say, if CD intersect the circle in E, F, each of the points of intersection might be the vertex of the required triangle. This supposes the length of the median Q to be greater than the altitude.

It may happen that the data of the problem are so related to one another that the resulting loci do not intersect. In this case the problem is impossible.

Obs In examples on the Intersection of Loci the student should make a point of investigating the relations which must exist among the data, in order that the problem may be possible, and he must observe that if under certain relations *two* solutions are possible, and under other relations no solution exists, there will always be some *intermediate* relation under which the two solutions combine in a single solution

EXAMPLES ON LOCI

1 Find the locus of a point which moves so that its distance (measured radially) from the circumference of a given circle is constant

2 A point P moves along a straight line RQ, find the position in which it is equidistant from two given points A and B

3 A and B are two fixed points within a circle find points on the circumference equidistant from A and B How many such points are there?

4 A point P moves along a straight line RQ, find the position in which it is equidistant from two given straight lines AB and CD

5 A and B are two fixed points 6 cm apart Find by the method of loci two points which are 4 cm distant from A, and 5 cm from B

6 AB and CD are two given straight lines Find points 3 cm. distant from AB, and 4 cm from CD How many solutions are there?

7 A straight rod of given length slides between two straight rulers placed at right angles to one another

Plot the locus of its middle point, and shew that this locus is the fourth part of the circumference of a circle [See Problem 10]

8 On a given base as hypotenuse right-angled triangles are described Find the locus of their vertices

9 A is a fixed point, and the point X moves on a fixed straight line BC

Plot the locus of P, the middle point of AX, and *prove* the locus to be a straight line parallel to BC

10 A is a fixed point, and the point X moves on the circumference of a given circle

Plot the locus of P, the middle point of AX, and *prove* that this locus is a circle [See Ex. 3, p 64.]

11. AB is a given straight line, and AX is the perpendicular drawn from A to any straight line passing through B. If BX revolve about B, find the locus of the middle point of AX.

12. Two straight lines OX, OY cut at right angles, and from P, a point within the angle XOY, perpendiculars PM, PN are drawn to OX, OY respectively. Plot the locus of P when

(i) PM + PN is constant (= 6 cm., say);

(ii) PM − PN is constant (= 3 cm., say).

And in each case give a theoretical proof of the result you arrive at experimentally.

13. Two straight lines OX, OY intersect at right angles at O; and from a movable point P perpendiculars PM, PN are drawn to OX, OY. Plot (without proof) the locus of P, when

(i) PM = 2 PN;

(ii) PM = 3 PN.

14. Find a point which is at a given distance from a given point, and is equidistant from two given parallel straight lines.

When does this problem admit of two solutions, when of one only, and when is it impossible?

15. S is a fixed point 2 inches distant from a given straight line MX. Find two points which are $2\frac{1}{4}$ inches distant from S, and also $2\frac{3}{4}$ inches distant from MX.

16. Find a series of points equidistant from a given point S and a given straight line MX. Draw a curve freehand passing through all the points so found.

17. On a given base construct a triangle of given altitude, having its vertex on a given straight line.

18. Find a point equidistant from the three sides of a triangle.

19. Two straight lines OX, OY cut at right angles; and Q and R are points in OX and OY respectively. Plot the locus of the middle point of QR, when

(i) OQ + OR = constant,

(ii) OQ − OR = constant.

20. S and S′ are two fixed points. Find a series of points P such that

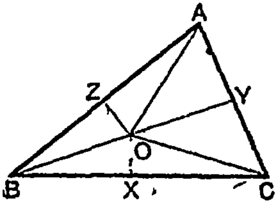
(i) SP + S′P = constant (say 3.5 inches);

(ii) SP − S′P = constant (say 1.5 inch).

In each case draw a curve freehand passing through all the points so found.

ON THE CONCURRENCE OF STRAIGHT LINES IN A TRIANGLE.

I. *The perpendiculars drawn to the sides of a triangle from their middle points are concurrent*



Let ABC be a △, and X, Y, Z the middle points of its sides

From Z and Y draw perps to AB, AC, meeting at O *Join* OX

It is required to prove that OX *is perp to* BC

Join OA, OB, OC

Proof Because YO bisects AC at right angles,
it is the locus of points equidistant from A and C,
OA=OC

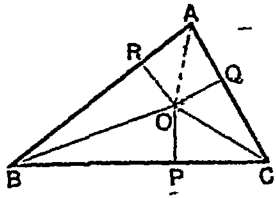
Again, because ZO bisects AB at right angles,
it is the locus of points equidistant from A and B,
OA=OB

Hence OB=OC
O is on the locus of points equidistant from B and C
that is, OX is perp to BC

Hence the perpendiculars from the mid points of the sides meet at O

Q E D

II *The bisectors of the angles of a triangle are concurrent*



Let ABC be a △ Bisect the ∠s ABC, BCA by straight lines which meet at O
Join AO

It is required to prove that AO *bisects the* ∠ BAC

From O draw OP, OQ, OR perp to the sides of the △

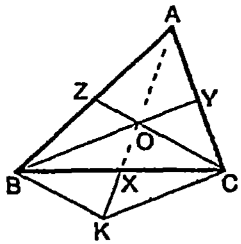
Proof Because BO bisects the ∠ ABC,
it is the locus of points equidistant from BA and BC,
OP=OR

Similarly CO is the locus of points equidistant from BC and CA,
OP=OQ.

Hence OR=OQ.
O is on the locus of points equidistant from AB and AC,
that is, OA is the bisector of the ∠ BAC

Hence the bisectors of the angles meet at O Q E D

III. *The medians of a triangle are concurrent*



Let ABC be a $\triangle$

Let BY and CZ be two of its medians, and let them intersect at O

Join AO,

and produce it to meet BC in X

It is required to shew that AX *is the remaining median of the* $\triangle$

Through C draw CK parallel to BY,

produce AX to meet CK at K

Join BK

Proof In the $\triangle$ AKC,

because Y is the middle point of AC, and YO is parallel to CK,

O is the middle point of AK *Theor* 22.

Again in the $\triangle$ ABK,

since Z and O are the middle points of AB, AK,

ZO is parallel to BK,

that is, OC is parallel to BK

the figure BKCO is a par^{m}

But the diagonals of a par^{m} bisect one another,

X is the middle point of BC

That is, AX is a median of the $\triangle$

Hence the three medians meet at the point O Q E D

DEFINITION The point of intersection of the medians is called the centroid of the triangle

COROLLARY *The three medians of a triangle cut one another at a point of trisection, the greater segment in each being towards the angular point*

For in the above figure it has been proved that

AO = OK,

also that OX is half of OK,

OX is half of OA

that is, OX is one third of AX

Similarly OY is one third of BY,

and OZ is one third of CZ Q E.D

By means of this Corollary it may be shewn that in any triangle the shorter median bisects the greater side

NOTE. It will be proved hereafter that *the perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent*

MISCELLANEOUS PROBLEMS

(*A theoretical proof is to be given in each case*)

1 A is a given point, and BC a given straight line From A draw a straight line to make with BC an angle equal to a given angle X

How many such lines can be drawn?

2 Draw the bisector of an angle AOB, without using the vertex O in your construction.

3 P is a given point within the angle AOB Draw through P a straight line terminated by OA and OB, and bisected at P

4 OA, OB, OC are three straight lines meeting at O Draw a transversal terminated by OA and OC, and bisected by OB

5 Through a given point A draw a straight line so that the part intercepted between two given parallels may be of given length

When does this problem admit of two solutions? When of only one? And when is it impossible?

6 In a triangle ABC inscribe a rhombus having one of its angles coinciding with the angle A.

7 Use the properties of an equilateral triangle to trisect a given straight line

(*Construction of Triangles*)

8 Construct a triangle, having given

(i) The middle points of the three sides

(ii) The lengths of two sides and of the median which bisects the third side

(iii) The lengths of one side and the medians which bisect the other two sides

(iv) The lengths of the three medians

PART II

ON AREAS

Definitions

1 The altitude (or height) of a parallelogram with reference to a given side as base, is the perpendicular distance between the base and the opposite side

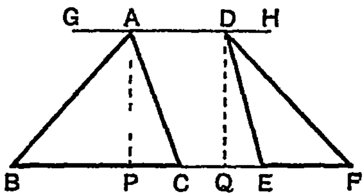
2 The altitude (or height) of a triangle with reference to a given side as base, is the perpendicular distance of the opposite vertex from the base

NOTE. It is clear that *parallelograms or triangles which are between the same parallels have the same altitude*

For let AP and DQ be the altitudes of the $\triangle^s$ ABC, DEF, which are between the same parallels BF, GH

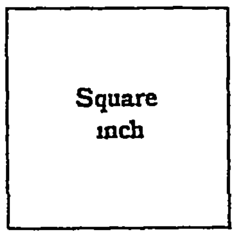
Then the fig APQD is evidently a rectangle,

$$AP = DQ$$



3 The area of a figure is the amount of surface contained within its bounding lines

4 A square inch is the area of a square drawn on a side one inch in length



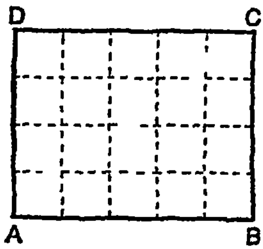
5 Similarly a square centimetre is the area of a square drawn on a side one centimetre in length

The terms *square yard, square foot, square metre* are to be understood in the same sense

6 Thus the unit of area is the area of a square on a side of unit length

THEOREM 23

Area of a rectangle *If the number of units in the length of a rectangle is multiplied by the number of units in its breadth, the product gives the number of square units in the area*



Let ABCD represent a rectangle whose length AB is 5 feet, and whose breadth AD is 4 feet

Divide AB into 5 equal parts, and BC into 4 equal parts, and through the points of division of each line draw parallels to the other

The rectangle ABCD is now divided into compartments, each of which represents one square foot

Now there are 4 rows, each containing 5 squares,

the rectangle contains 5×4 square feet

Similarly, if the length $= a$ linear units, and the breadth $= b$ linear units

the rectangle contains ab *units of* area.

And if each side of a square $= a$ linear units,

the square contains a² *units of* area

These statements may be thus abridged -

$$\text{the area of a rectangle} = \text{length} \times \text{breadth} \qquad (\text{i}),$$
$$\text{the area of a square} = (\text{side})^2 \qquad (\text{ii})$$

Q E D

COROLLARIES (i) *Rectangles which have equal lengths and equal breadths have equal areas*

(ii) *Rectangles which have equal areas and equal lengths have also equal breadths*

NOTATION

The rectangle ABCD is said to be contained by AB, AD, for these adjacent sides fix its size and shape

A rectangle whose adjacent sides are AB, AD is denoted by *rect* AB, AD, or simply AB × AD

A square drawn on the side AB is denoted by *sq on* AB, or AB^2

EXERCISES

(On Tables of Length and Area.)

1 Draw a figure to shew *why*

(i) 1 sq yard $= 3^2$ sq feet
(ii) 1 sq foot $= 12^2$ sq inches
(iii) 1 sq cm $= 10^2$ sq mm

2. Draw a figure to shew that the square on a straight line is four times the square on half the line

3 Use squared paper to shew that the square on $1'' = 10^2$ times the square on $0{\cdot}1''$

4 If $1''$ represents 5 miles, what does an area of 6 square inches represent?

EXTENSION OF THEOREM 23

The proof of Theorem 23 here given supposes that the length and breadth of the given rectangle are expressed by *whole numbers*, but the formula holds good when the length and breadth are fractional

This may be illustrated thus

Suppose the length and breadth are 3·2 cm and 2·4 cm, we shall shew that the area is $(3{\cdot}2 \times 2{\cdot}4)$ sq cm

For

$$\text{length} = 3{\cdot}2 \text{ cm} = 32 \text{ mm}$$
$$\text{breadth} = 2{\cdot}4 \text{ cm} = 24 \text{ mm}$$
$$\text{area} = (32 \times 24) \text{ sq mm} = \frac{32 \times 24}{10^2} \text{ sq cm}$$
$$= (3{\cdot}2 \times 2{\cdot}4) \text{ sq cm}$$

EXERCISES

(*On the Area of a Rectangle.*)

Draw on squared paper the rectangles of which the length (a) and breadth (b) are given below Calculate the areas, and verify by the actual counting of squares

1 $a=2''$, $b=3''$ 2 $a=1\ 5''$, $b=4''$

3 $a=0{\cdot}8''$, $b=3\ 5''$ 4 $a=2\ 5''$, $b=1\ 4''$

5 $a=2{\cdot}2''$, $b=1\ 5''$ 6 $a=1\ 6''$, $b=2\ 1''$

Calculate the areas of the rectangles in which

7 $a=18$ metres, $b=11$ metres. 8 $a=7$ ft $b=72$ in.

9 $a=2\ 5$ km, $b=4$ metres. 10 $a=\frac{1}{4}$ mile, $b=1$ inch

11 The area of a rectangle is 30 sq cm, and its length is 6 cm Find the breadth Draw the rectangle on squared paper, and verify your work by counting the squares

12 Find the length of a rectangle whose area is 3 9 sq in, and breadth 1 5″ Draw the rectangle on squared paper, and verify your work by counting the squares

13 (i) When you treble the length of a rectangle without altering its breadth, how many times do you multiply the area?

(ii) When you treble both length and breadth, how many times do you multiply the area?

Draw a figure to illustrate your answers, and state a general rule

14 In a plan of a rectangular garden the length and breadth are 3 6″ and 2 5′, one inch standing for 10 yards Find the area of the garden

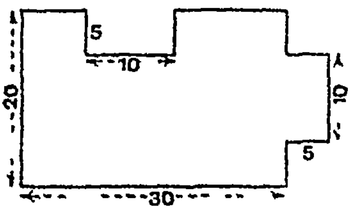
If the area is increased by 300 sq yds, the breadth remaining the same, what will the new length be? And how many inches will represent it on your plan?

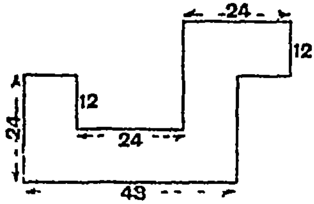
15 Find the area of a rectangular enclosure of which a plan (scale 1 cm to 20 metres) measures 6 5 cm by 4 5 cm

16 The area of a rectangle is 1440 sq yds If in a plan the sides of the rectangle are 3 2 cm and 4 5 cm, on what scale is the plan drawn?

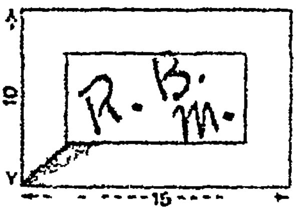
17 The area of a rectangular field is 52000 sq ft On a plan of this, drawn to the scale of 1″ to 100 ft, the length is 3·25″ What is the breadth?

Calculate the areas of the enclosures of which plans are given below All the angles are right angles, and the dimensions are marked in feet

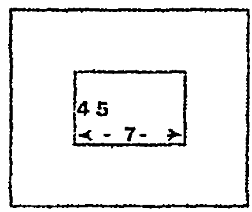
18 

19 

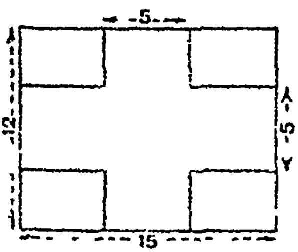
Calculate the areas represented by the shaded parts of the following plans The dimensions are marked in feet

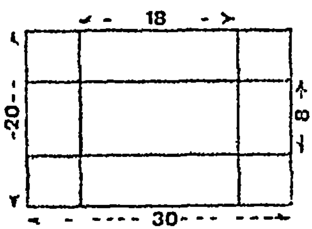
20 

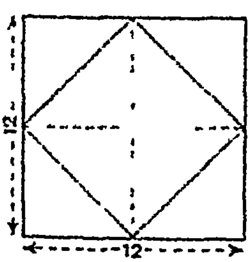
Width of shaded border uniform 2½ ft.

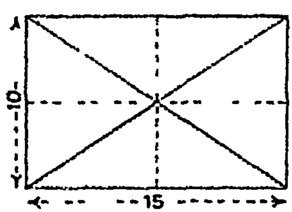
21 

Width of shaded border uniform 4 ft.

22 

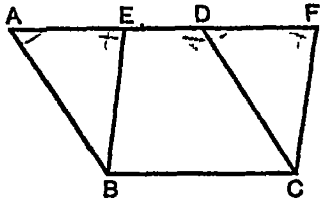
23 

24 

25 

THEOREM 24 [Euclid I 35]

Parallelograms on the same base and between the same parallels are equal in area



Let the par^{ms} ABCD, EBCF be on the same base BC, and between the same par^{ls} BC, AF

It is required to prove that

the par^{m} ABCD = *the par^{m}* EBCF *in area*

Proof In the $\triangle^s$ FDC, EAB,

because $\begin{cases} \text{DC} = \text{the opp side AB}, & \textit{Theor } 21 \\ \text{the ext } \angle\,\text{FDC} = \text{the int opp } \angle\,\text{EAB}, & \textit{Theor } 14 \\ \text{the int. } \angle\,\text{DFC} = \text{the ext } \angle\,\text{AEB}, & \end{cases}$

the $\triangle$ FDC = the $\triangle$ EAB *Theor* 17

Now, if from the whole fig ABCF the $\triangle$ FDC is taken, the remainder is the par^{m} ABCD

And if from the whole fig ABCF the $\triangle$ EAB is taken, the remainder is the par^{m} EBCF

these remainders are equal,

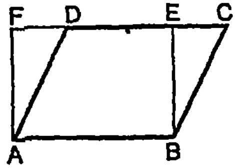
that is, the par^{m} ABCD = the par^{m} EBCF Q E D

EXERCISE

In the above diagram the sides AD, EF overlap Draw diagrams in which (i) these sides do not overlap, (ii) the ends E and D coincide

Go through the proof with these diagrams, and ascertain if it applies to them without change.

THE AREA OF A PARALLELOGRAM.



Let ABCD be a parallelogram, and ABEF the rectangle on the same base AB and of the same altitude BE Then by Theorem 24,

$$\textit{area of par}^{m}\ ABCD = \text{area of rect } ABEF$$
$$= AB \times BE$$
$$= \textit{base} \times \textit{altitude}$$

COROLLARY Since the area of a parallelogram depends only on its base and altitude, it follows that

Parallelograms on equal bases and of equal altitudes are equal in area

EXERCISES

(Numerical and Graphical)

1 Find the area of parallelograms in which
(i) the base = 5 5 cm , and the height = 4 cm
(ii) the base = 2 4″, and the height = 1 5″

2 Draw a parallelogram ABCD having given AB = $2\frac{1}{2}$″, AD = $1\frac{1}{2}$″, and the $\angle A = 65°$ Draw and measure the perpendicular from D on AB, and hence calculate the approximate area Why *approximate*?

Again calculate the area from the length of AD and the perpendicular on it from B Obtain the average of the two results

3 Two adjacent sides of a parallelogram are 30 metres and 25 metres, and the included angle is 50° Draw a plan, 1 cm representing 5 metres, and by measuring each altitude, make two independent calculations of the area Give the average result

4. The area of a parallelogram ABCD is 4 2 sq in , and the base AB is 2 8″ Find the height. If AD = 2″, draw the parallelogram

5 Each side of a rhombus is 2″, and its area is 3 36 sq in Calculate an altitude Hence draw the rhombus, and measure one of its acute angles

THEOREM 25

The Area of a Triangle *The area of a triangle is half the area of the rectangle on the same base and having the same altitude*

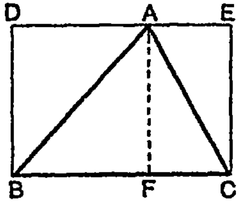


Fig 1

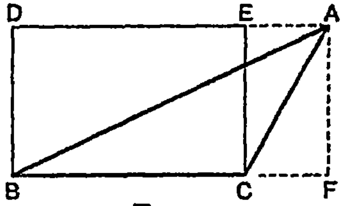


Fig 2

Let ABC be a triangle, and BDEC a rectangle on the same base BC and with the same altitude AF

It is required to prove that the △ ABC *is half the rectangle* BDEC

Proof Since AF is perp to BC, each of the figures DF, EF is a rectangle

Because the diagonal AB bisects the rectangle DF,
the △ ABF is half the rectangle DF

Similarly, the △ AFC is half the rectangle FE

adding these results in Fig 1, and taking the difference in Fig 2,

the △ ABC is half the rectangle BDEC

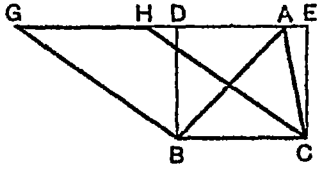
Q E D

COROLLARY *A triangle is half any parallelogram on the same base and between the same parallels*

For the △ ABC is half the rect BCED

And the rect BCED = any par^{m} BCHG on the same base and between the same par^{ls}

the △ ABC is half the par^{m} BCHG



THE AREA OF A TRIANGLE

If BC and AF respectively contain a units and p units of length, the rectangle BDEC contains ap units of area

the area of the $\triangle$ ABC $= \frac{1}{2}ap$ units of area

This result may be stated thus

Area of a Triangle $= \frac{1}{2}$ *base × altitude*

EXERCISES ON THE AREA OF A TRIANGLE.

(*Numerical and Graphical*)

1 Calculate the areas of the triangles in which

(i) the base = 24 ft, the height = 15 ft

(ii) the base = 4 8", the height = 3 5"

(iii) the base = 160 metres, the height = 125 metres

2 Draw triangles from the following data In each case draw and measure the altitude with reference to a given side as base hence calculate the approximate area

(i) $a = 8\ 4$ cm, $b = 6\ 8$ cm, $c = 4{\cdot}0$ cm

(ii) $b = 5\ 0$ cm, $c = 6\ 8$ cm, $A = 65°$

(iii) $a = 6\ 5$ cm, $B = 52°$, $C = 76$

3 ABC is a triangle *right angled* at C shew that its area $= \frac{1}{2}$ BC × CA

Given $a = 6$ cm $b = 5$ cm, calculate the area

Draw the triangle and measure the hypotenuse c, draw and measure the perpendicular from C on the hypotenuse hence calculate the approximate area

Note the error in your approximate result, and express it as a percentage of the true value

4 Repeat the whole process of the last question for a right-angled triangle ABC, in which $a = 2\ 8"$ and $b = 4\ 5"$, C being the right angle as before

5 In a triangle, given

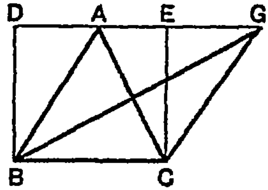
(i) Area = 80 sq in, base = 1 ft 8 in, calculate the altitude

(ii) Area = 10 4 sq cm, altitude = 1 6 cm, calculate the base

6 Construct a triangle ABC, having given $a = 3\ 0"$, $b = 2\ 8"$, $c = 2\ 6"$ Draw and measure the perpendicular from A on BC, hence calculate the approximate area

THEOREM 26 [Euclid I 37]

Triangles on the same base and between the same parallels (hence, of the same altitude) are equal in area



Let the $\triangle^s$ ABC, GBC be on the same base BC and between the same par^{ls} BC, AG

It is required to prove that the $\triangle$ ABC = *the* $\triangle$ GBC *in area*

Proof If BCED is the rectangle on the base BC, and between the same parallels as the given triangles,

the $\triangle$ ABC is half the rect BCED, *Theor* 25

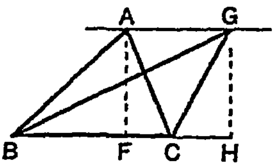
also the $\triangle$ GBC is half the rect BCED,

the $\triangle$ ABC = the $\triangle$ GBC Q E D

Similarly, *triangles on equal bases and of equal altitudes are equal in area*

THEOREM 27 [Euclid I 39]

If two triangles are equal in area, and stand on the same base and on the same side of it, they are between the same parallels



Let the $\triangle^s$ ABC, GBC, standing on the same base BC, be equal in area, and let AF and GH be their altitudes

It is required to prove that AG *and* BC *are par^l*

Proof The $\triangle$ ABC is half the rectangle contained by BC and AF,

and the $\triangle$ GBC is half the rectangle contained by BC and GH,

the rect BC, AF = the rect BC, GH,

AF = GH *Theor* 23, *Cor* 2

Also AF and GH are par^l,

hence AG and FH, that is BC, are par^l Q E.D

EXERCISES ON THE AREA OF A TRIANGLE

(Theoretical.)

1. ABC is a triangle and XY is drawn parallel to the base BC, cutting the other sides at X and Y. Join BY and CX, and shew that

(i) the $\triangle$ XBC = the $\triangle$ YBC,
(ii) the $\triangle$ BXY = the $\angle$ CXY,
(iii) the $\triangle$ ABY = the $\triangle$ ACX.

If BY and CX cut at K, shew that

(iv) the $\angle$ BKX = the $\triangle$ CKY.

2. Shew that a median of a triangle divides it into two parts of equal area.

How would you divide a triangle into *three* equal parts by straight lines drawn from its vertex?

3. Prove that a parallelogram is divided by its diagonals into four triangles of equal area.

4. ABC is a triangle whose base BC is bisected at X. If Y is any point in the median AX, shew that

the $\angle$ ABY = the $\triangle$ ACY in area.

5. ABCD is a parallelogram, and BP, DQ are the perpendiculars from B and D on the diagonal AC.

Shew that BP = DQ.

Hence if X is any point in AC, or AC produced,

prove (i) the $\triangle$ ADX = the $\angle$ ABX,
(ii) the $\angle$ CDX = the $\angle$ CBX.

6. Prove by means of Theorems 26 and 27 that *the straight line joining the middle points of two sides of a triangle is parallel to the third side.*

7. The straight line which joins the middle points of the oblique sides of a trapezium is parallel to each of the parallel sides.

8. ABCD is a parallelogram and X, Y are the middle points of the sides AD, BC; if Z is any point in XY, or XY produced, shew that the triangle AZB is one quarter of the parallelogram ABCD.

9. If ABCD is a parallelogram, and X, Y any points in DC and AD respectively: shew that the triangles AXB, BYC are equal in area.

10. ABCD is a parallelogram, and P is any point within it; shew that the sum of the triangles PAB, PCD is equal to half the parallelogram.

EXERCISES ON THE AREA OF A TRIANGLE

(Numerical and Graphical)

1 The sides of a triangular field are 370 yds, 200 yds, and 190 yds Draw a plan (scale 1" to 100 yards) Draw and measure an altitude, hence calculate the approximate area of the field in square yards

2 Two sides of a triangular enclosure are 124 metres and 144 metres respectively, and the included angle is observed to be 45° Draw a plan (scale 1 cm to 20 metres) Make any necessary measurement, and calculate the approximate area

3 In a triangle ABC, given that the area = 6 6 sq cm, and the base BC = 5 5 cm, find the altitude Hence determine the locus of the vertex A.

If in addition to the above data, BA = 2 6 cm, construct the triangle, and measure CA

4 In a triangle ABC, given *area* = 3 06 sq in, and $a = 3\ 0''$ Find the altitude, and the locus of A Given C = 68°, construct the triangle, and measure b

5 ABC is a triangle in which BC, BA have constant lengths 6 cm and 5 cm If BC is fixed, and BA revolves about B, trace the changes in the area of the triangle as the angle B increases from 0° to 180°

Answer this question by drawing a series of triangles, increasing B by increments of 30° Find the area in each case and tabulate the results

(Theoretical)

6 If two triangles have two sides of one respectively equal to two sides of the other, and the angles contained by those sides *supplementary*, shew that the triangles are equal *in area* Can such triangles ever be *identically* equal?

7 Shew how to draw on the base of a given triangle an isosceles triangle of equal area.

8 If the middle points of the sides of a quadrilateral are joined in order, prove that the *parallelogram* so formed [see Ex 7, p 64], is half the quadrilateral

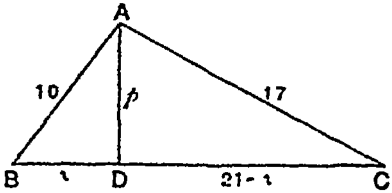
9 ABC is a triangle, and R, Q the middle points of the sides AB, AC, shew that if BQ and CR intersect in X, the triangle BXC is equal to the quadrilateral AQXR

10 Two triangles of equal area stand on the same base but on opposite sides of it shew that the straight line joining their vertices is bisected by the base, or by the base produced

[The method given below may be omitted from a first course. In any case it must be postponed till Theorem 29 has been read.]

The Area of a Triangle. *Given the three sides of a triangle, to calculate the area.*

EXAMPLE. *Find the area of a triangle whose sides measure* 21 m., 17 m., *and* 10 m.



Let ABC represent the given triangle.

Draw AD perp. to BC, and denote AD by p.

We shall first find the length of BD.

Let BD $= x$ metres: then DC $= 21 - x$ metres.

From the right-angled $\triangle$ ADB, we have by Theorem 29,

$$AD^2 = AB^2 - BD^2 = 10^2 - x^2.$$

And from the right-angled $\triangle$ ADC,

$$AD^2 = AC^2 - DC^2 = 17^2 - (21 - x)^2.$$

$$\therefore \quad 10^2 - x^2 = 17^2 - (21 - x)^2,$$

or $$100 - x^2 = 289 - 441 + 42x - x^2,$$

whence $$x = 6.$$

Again, $$AD^2 = AB^2 - BD^2,$$

or $$p^2 = 10^2 - 6^2 = 64,$$

$$\therefore \quad p = 8.$$

Now *Area of triangle* $= \frac{1}{2}$ *base* $\times$ *altitude*

$= (\frac{1}{2} \times 21 \times 8)$ sq. m. $= 84$ sq. m.

EXERCISES.

Find by the above method the area of the triangles, whose sides are as follows:

1. 20 ft., 13 ft., 11 ft.
2. 15 yds., 14 yds., 13 yds.
3. 21 m., 20 m., 13 m.
4. 30 cm., 25 cm., 11 cm.
5. 37 ft., 30 ft., 13 ft.
6. 51 m., 37 m., 20 m.
7. If the given sides are a, b and c units in length, prove

(i) $x = \dfrac{a^2 + c^2 - b^2}{2a}$, (ii) $p^2 = c^2 - \left\{\dfrac{a^2 + c^2 - b^2}{2a}\right\}^2$,

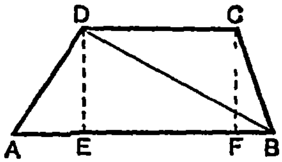
(iii) $\triangle = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$.

THE AREA OF QUADRILATERALS

Theorem 28

To find the area of (i) *a trapezium*
(ii) *any quadrilateral*

(i) Let ABCD be a trapezium, having the sides AB, CD parallel. Join BD, and from C and D draw perpendiculars CF, DE to AB



Let the parallel sides AB, CD measure a and b units of length, and let the height CF contain h units

Then the area of ABCD $= \triangle ABD + \triangle DBC$

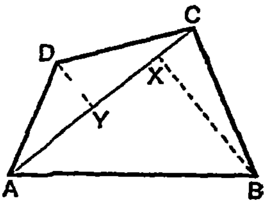
$$= \frac{1}{2} AB \cdot DE + \frac{1}{2} DC \cdot CF$$

$$= \frac{1}{2} ah + \frac{1}{2} bh = \frac{h}{2}(a+b)$$

That is,

the area of a trapezium $= \frac{1}{2}$ *height* $\times$ *(the sum of the parallel sides)*

(ii) Let ABCD be any quadrilateral Draw a diagonal AC, and from B and D draw perpendiculars BX, DY to AC These perpendiculars are called offsets



If AC contains d units of length, and BX, DY p and q units respectively,

the area of the quad[l] ABCD $= \triangle ABC + \triangle ADC$

$$= \frac{1}{2} AC \cdot BX + \frac{1}{2} AC \cdot DY$$

$$= \frac{1}{2} dp + \frac{1}{2} dq = \frac{1}{2} d(p+q)$$

That is to say,

the area of a quadrilateral $= \frac{1}{2}$ *diagonal* $\times$ *(sum of offsets).*

EXERCISES

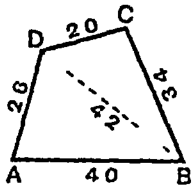
(Numerical and Graphical)

1 Find the area of the trapezium in which the two parallel sides are 4·7" and 3·3", and the height 1·5".

2 In a quadrilateral ABCD, the diagonal AC = 17 feet, and the offsets from it to B and D are 11 feet and 9 feet. Find the area.

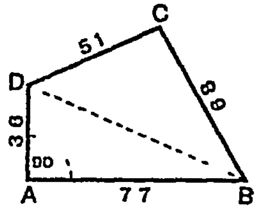
3 In a plan ABCD of a quadrilateral enclosure, the diagonal AC measures 8·2 cm, and the offsets from it to B and D are 3·4 cm and 2·6 cm respectively. If 1 cm in the plan represents 5 metres, find the area of the enclosure.

4 Draw a quadrilateral ABCD from the adjoining rough plan, the dimensions being given in inches.



Draw and measure the offsets to A and C from the diagonal BD, and hence calculate the area of the quadrilateral.

5 Draw a quadrilateral ABCD from the details given in the adjoining plan. The dimensions are to be in centimetres.



Make any necessary measurements of your figure, and calculate its area.

6 Draw a trapezium ABCD from the following data. AB and CD are the parallel sides. AB = 1", AD = BC = 2", the ∠A = the ∠B = 60°.

Make any necessary measurements, and calculate the area.

7 Draw a trapezium ABCD in which AB and CD are the parallel sides, and AB = 9 cm, CD = 3 cm, and AD = BC = 5 cm.

Make any necessary measurement, and calculate the area.

8 From the formula *area of quad' = ½ diag × (sum of offsets)* shew that, if the diagonals are *at right angles*,

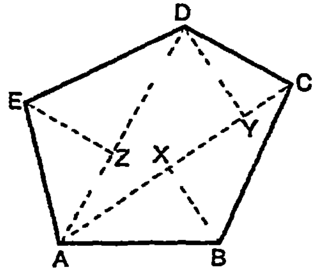
area = ½ (product of diagonals)

9 Given the lengths of the diagonals of a quadrilateral, and the angle between them, prove that the area is the same wherever they intersect.

THE AREA OF ANY RECTILINEAL FIGURE

1st METHOD A rectilineal figure may be divided into triangles whose areas can be separately calculated from suitable measurements The sum of these areas will be the area of the given figure

Example. The measurements required to find the area of the figure ABCDE are AC, AD, and the offsets BX, DY, EZ.

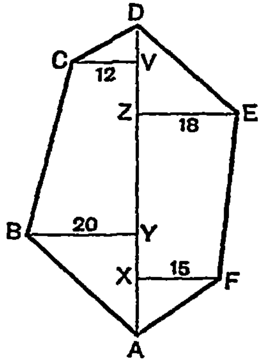


2nd METHOD The area of a rectilineal figure is also found by taking a base-line (AD in the diagram below) and offsets from it These divide the figure into *right-angled* triangles and *right-angled* trapeziums, whose areas may be found after measuring the offsets and the various sections of the base-line

Example. Find the area of the enclosure ABCDEF from the plan and measurements tabulated below

	YARDS	
	AD = 56	
VC = 12	AV = 50	
	AZ = 40	ZE = 18
YB = 20	AY = 18	
	AX = 10	XF = 15

The measurements are made from A along the base line to the points from which the offsets spring



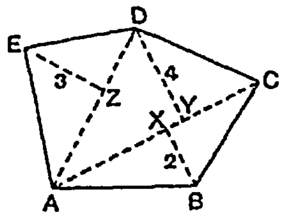
			Sq yds.	Sq yds
Here	△ AXF = ½	AX × XF	= ½ × 10 × 15 =	75
	△ AYB = ½	AY × YB	= ½ × 18 × 20 =	180
	△ DZE = ½	DZ × ZE	= ½ × 16 × 18 =	144
	△ DVC = ½	DV × VC	= ½ × 6 × 12 =	36
	trapm XFEZ = ½	XZ × (XF + ZE)	= ½ × 30 × 33 =	495
	trapm YBCV = ½	YV × (YB + VC)	= ½ × 32 × 32 =	512

∴ by addition, the fig ABCDEF = 1442 sq yds

EXERCISES

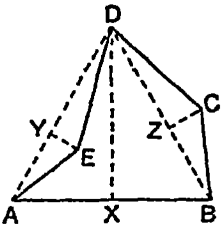
1 Calculate the areas of the figures (i) and (ii) from the plans and dimensions (in cms.) given below

(i)



AC=6 cm, AD=5 cm
Lengths of offsets figured in diagram

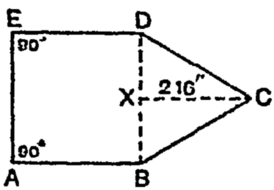
(ii)



AB=BD=DA=6 cm
EY=CZ=1 cm
DX=5·2 cm

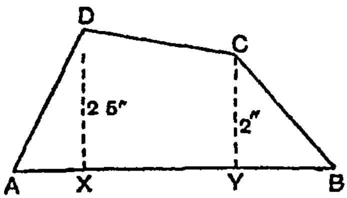
2. Draw full size the figures whose plans and dimensions are given below, and calculate the area in each case

(i)



The fig is equilateral each side to be $2\frac{1}{2}''$

(ii)

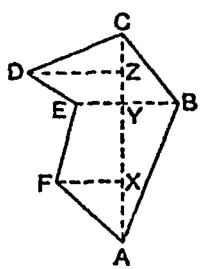


AX=$1\frac{1}{4}''$, XY=$2\frac{3}{4}''$, YB=$1\frac{1}{4}''$

3 Find the area of the figure ABCDEF from the following measurements and draw a plan in which 1 cm represents 20 metres

	Metres	
	to C	
	180	
80 to D	150	
40 to E	120	50 to B
60 to F	50	
	From A	

The Plan



EXERCISES ON QUADRILATERALS

(*Theoretical*)

1 ABCD is a rectangle, and PQRS the figure formed by joining in order the middle points of the sides

Prove (i) that PQRS is a rhombus,

(ii) that the area of PQRS is half that of ABCD

Hence shew that *the area of a rhombus is half the product of its diagonals*

Is this true of any quadrilateral whose diagonals cut at right angles? Illustrate your answer by a diagram

2. Prove that a parallelogram is bisected by any straight line which passes through the middle point of one of its diagonals

Hence shew how a parallelogram ABCD may be bisected by a straight line drawn

(i) through a given point P,

(ii) perpendicular to the side AB,

(iii) parallel to a given line QR

3 In the trapezium ABCD, AB is parallel to DC, and X is the middle point of BC Through X draw PQ parallel to AD to meet AB and DC produced at P and Q. Then prove

(i) trapezium ABCD = par^{m} APQD

(ii) trapezium ABCD = twice the $\triangle$AXD

(*Graphical*)

4 The diagonals of a quadrilateral ABCD cut at right angles, and measure 3 0" and 2 2" respectively Find the area

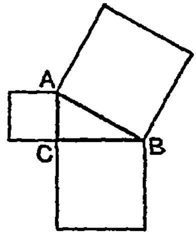
Shew by a figure that the area is the same wherever the diagonals cut, so long as they are at right angles.

5 In the parallelogram ABCD, AB = 8 0 cm, AD = 3·2 cm, and the perpendicular distance between AB and DC = 3·0 cm Draw the parallelogram Calculate the distance between AD and BC, and check your result by measurement

6 One side of a parallelogram is 2 5", and its diagonals are 3 4" and 2 4" Construct the parallelogram, and, after making any necessary measurement, calculate the area

7 ABCD is a parallelogram on a fixed base AB and of constant area Find the locus of the intersection of its diagonals

EXERCISES LEADING TO THEOREM 29



In the adjoining diagram, ABC is a triangle *right angled* at C, and squares are drawn on the three sides. Let us compare the area of the square on the hypotenuse AB with the *sum* of the squares on the sides AC, CB which contain the right angle

1 Draw the above diagram, making AC=3 cm, and BC=4 cm;

Then the area of the square on $AC=3^2$, or 9 sq cm.
and the square on $BC=4^2$, or 16 sq cm

the sum of the squares on AC, BC= 25 sq cm

Now *measure* AB, hence calculate the area of the square on AB, and compare the result with the *sum* already obtained

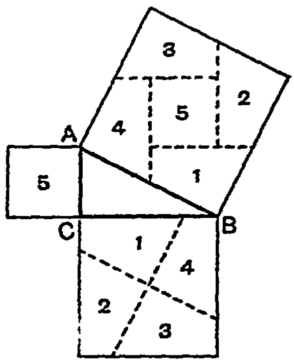
2. Repeat the process of the last exercise, making AC=1 0″, and BC=2 4″

3 If $a=15$, $b=8$, $c=17$, shew arithmetically that $c^2=a^2+b^2$

Now draw on squared paper a triangle ABC, whose sides a, b, and c are 15, 8, and 17 units of length, and *measure the angle* ACB

4 Take any triangle ABC, right-angled at C, and draw squares on AC, CB, and on the hypotenuse AB

Through the mid-point of the square on CB (i e the intersection of the diagonals) draw lines parallel and perpendicular to the hypotenuse, thus dividing the square into four *congruent* quadrilaterals. These together with the square on AC, will be found exactly to fit into the square on AB, in the way indicated by corresponding numbers



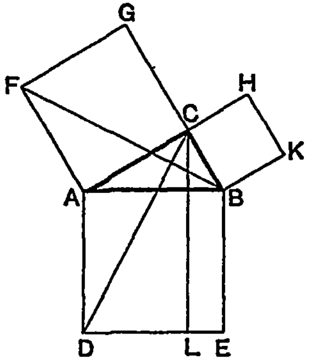
These experiments point to the conclusion that

In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides

A formal proof of this theorem is given on the next page

THEOREM 29 [Euclid I 47]

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides



Let ABC be a right-angled △, having the angle ACB a rt ∠

It is required to prove that the square on the hypotenuse AB = *the sum of the squares on* AC, CB

On AB describe the sq ADEB, and on AC, CB describe the sqq ACGF, CBKH

Through C draw CL par[l] to AD or BE

Join CD, FB

Proof Because each of the ∠[s] ACB, ACG is a rt ∠,
BC and CG are in the same st line

Now the rt. ∠ BAD = the rt ∠ FAC,
add to each the ∠ CAB
then the whole ∠ CAD = the whole ∠ FAB

Then in the △[s] CAD, FAB,

because $\begin{cases} \text{CA} = \text{FA}, \\ \text{AD} = \text{AB}, \\ \text{and the included } \angle\,\text{CAD} = \text{the included } \angle\,\text{FAB}, \end{cases}$

the △ CAD = the △ FAB *Theor* 4

Now the rect. AL is double of the $\triangle$ CAD, being on the same base AD, and between the same par[ls] AD, CL.

And the sq GA is double of the $\triangle$ FAB, being on the same base FA, and between the same par[ls] FA, GB

the rect AL = the sq GA

Similarly by joining CE, AK, it can be shewn that

the rect BL = the sq HB

the whole sq AE = the sum of the sqq GA, HB

that is, the square on the hypotenuse AB = the sum of the squares on the two sides AC, CB

Q E D

Obs This is known as the Theorem of Pythagoras The result established may be stated as follows

$$AB^2 = BC^2 + CA^2$$

That is, if a and b denote the lengths of the sides containing the right angle, and if c denotes the hypotenuse,

$$c^2 = a^2 + b^2$$

Hence $\qquad a^2 = c^2 - b^2$, and $b^2 = c^2 - a^2$

NOTE 1 The following important results should be noticed

If CL and AB intersect in O, it has been shewn in the course of the proof that

the sq GA = the rect AL,
that is, AC^2 = the rect contained by AB AO (i)
Also the sq HB = the rect BL,
that is, BC^2 = the rect contained by BA, BO (ii)

NOTE 2 It can be proved by superposition that *squares standing on equal sides are equal in area*

Hence we conclude, conversely,

If two squares are equal in area they stand on equal sides

EXPERIMENTAL PROOFS OF PYTHAGORAS'S THEOREM.

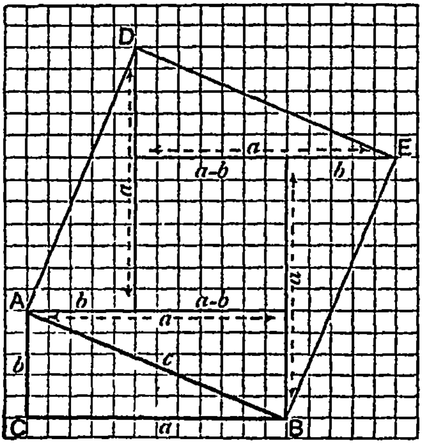
I Here ABC is the given rt angled $\triangle$, and ABED is the square on the hypotenuse AB

By drawing lines par^l to the sides BC, CA, it is easily seen that the sq BD is divided into 4 rt angled $\triangle$^s, each identically equal to ABC, together with a central square

Hence

sq on hypotenuse c = 4 rt $\angle$^d $\triangle$^s + the central square

$$= 4 \cdot \tfrac{1}{2}ab + (a-b)^2$$
$$= 2ab + a^2 - 2ab + b^2$$
$$= a^2 + b^2$$

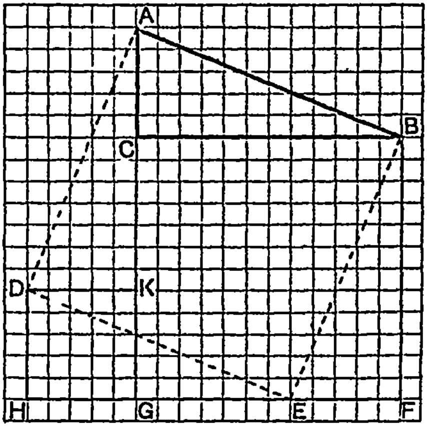


II Here ABC is the given rt angled $\triangle$, and the figs CF, HK are the sqq on CB, CA placed side by side

FE is made equal to DH or CA, and the two sqq CF, HK are cut along the lines BE, ED

Then it will be found that the $\triangle$ DHE may be placed so as to fill up the space ACB, and the $\triangle$ BFE may be made to fill the space AKD

Hence the two sqq CF, HK may be fitted together so as to form the single fig ABED, which will be found to be a perfect square, namely the square on the hypotenuse AB



EXERCISES

(*Numerical and Graphical.*)

1. Draw a triangle ABC, right angled at C, having given

(i) $a=3$ cm, $b=4$ cm;
(ii) $a=2·5$ cm, $b=6·0$ cm;
(iii) $a=1·2''$, $b=3·5''$.

In each case calculate the length of the hypotenuse c, and verify your result by measurement.

2. Draw a triangle ABC, right angled at C, having given

(i) $c=3·4''$, $a=3·0''$. [See Problem 10.]
(ii) $c=5·3$ cm, $b=4·5$ cm.

In each case calculate the remaining side, and verify your result by measurement.

(*The following examples are to be solved by calculation; but in each case a plan should be drawn on some suitable scale, and the calculated result verified by measurement.*)

3. A ladder whose foot is 9 feet from the front of a house reaches to a window sill 40 feet above the ground. What is the length of the ladder?

4. A ship sails 33 miles due South, and then 56 miles due West. How far is it then from its starting point?

5. Two ships are observed from a signal station to bear respectively N.E. 6·0 km. distant, and N.W. 1·1 km. distant. How far are they apart?

6. A ladder 65 feet long reaches to a point in the face of a house 63 feet above the ground. How far is the foot from the house?

7. B is due East of A, but at an unknown distance. C is due South of B, and distant 55 metres. AC is known to be 73 metres. Find AB.

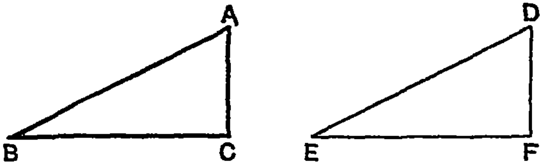
8. A man travels 27 miles due South, then 24 miles due West, finally 20 miles due North. How far is he from his starting point?

9. From A go West 25 metres, then North 60 metres, then East 80 metres, finally South 12 metres. How far are you then from A?

10. A ladder 50 feet long is placed so as to reach a window 48 feet high; and on turning the ladder over to the other side of the street, it reaches a point 14 feet high. Find the breadth of the street.

THEOREM 30 [Euclid I 48]

If the square described on one side of a triangle is equal to the sum of the squares described on the other two sides, then the angle contained by these two sides is a right angle



Let ABC be a triangle in which
the sq on AB = the sum of the sqq on BC, CA.

It is required to prove that ACB *is a right angle*

Make EF equal to BC
Draw FD perp^{r} to EF, and make FD equal to CA.
Join ED

Proof Because EF = BC,
the sq on EF = the sq on BC
And because FD = CA,
the sq on FD = the sq on CA
Hence the sum of the sqq on EF, FD = the sum of the sqq on BC, CA
But since EFD is a rt $\angle$,
the sum of the sqq on EF, FD = the sq on DE *Theor* 29
And, by hypothesis, the sqq on BC, CA = the sq on AB
the sq on DE = the sq on AB
DE = AB

Then in the $\triangle^{s}$ ACB, DFE,

because $\begin{cases} AC = DF, \\ CB = FE, \\ \text{and } AB = DE, \end{cases}$

the $\angle$ ACB = the $\angle$ DFE *Theor* 7
But, by construction, DFE is a right angle
the $\angle$ ACB is a right angle

Q E D

EXERCISES ON THEOREMS 29, 30

(Theoretical.)

1 Shew that the square on the diagonal of a given square is double of the given square

2 In the $\triangle ABC$, AD is drawn perpendicular to the base BC If the side c is greater than b,

$$\text{shew that } c^2 - b^2 = BD^2 - DC^2$$

3 If from any point O within a triangle ABC, perpendiculars OX, OY, OZ are drawn to BC, CA, AB respectively shew that

$$AZ^2 + BX^2 + CY^2 = AY^2 + CX^2 + BZ^2$$

4 ABC is a triangle right angled at A and the sides AB, AC are intersected by a straight line PQ, and BQ, PC are joined Prove that

$$BQ^2 + PC^2 = BC^2 + PQ^2$$

5 In a right-angled triangle four times the sum of the squares on the medians drawn from the acute angles is equal to five times the square on the hypotenuse

6 Describe a square equal to the sum of two given squares

7 Describe a square equal to the difference between two given squares.

8 Divide a straight line into two parts so that the square on one part may be twice the square on the other

9 Divide a straight line into two parts such that the sum of their squares shall be equal to a given square

(Numerical and Graphical.)

10 Determine which of the following triangles are right-angled

(i) $a = 14$ cm, $b = 48$ cm, $c = 50$ cm,
(ii) $a = 40$ cm, $b = 10$ cm, $c = 41$ cm
(iii) $a = 20$ cm, $b = 99$ cm, $c = 101$ cm

11 ABC is an isosceles triangle right angled at C, deduce from Theorem 29 that

$$AB^2 = 2AC^2$$

Illustrate this result graphically by drawing both diagonals of the square on AB, and one diagonal of the square on AC

If AC = BC = 2", find AB to the nearest hundredth of an inch, and verify your calculation by actual construction and measurement

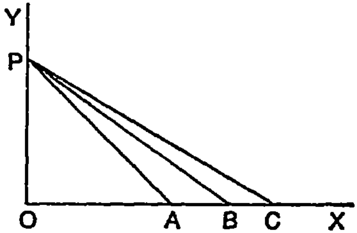
12. Draw a square on a diagonal of 6 cm Calculate, and also measure, the length of a side. Find the area

PROBLEM 16

To draw squares whose areas shall be respectively twice, three-times, four-times, , that of a given square.

Hence find graphically approximate values of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$,

Take OX, OY at right angles to one another, and from them mark off OA, OP, each *one* unit of length Join PA



Then $PA^2 = OP^2 + OA^2 = 1 + 1 = 2$

$$PA = \sqrt{2}$$

From OX mark off OB equal to PA, and join PB,
then $PB^2 = OP^2 + OB^2 = 1 + 2 = 3$

$$PB = \sqrt{3}$$

From OX mark off OC equal to PB, and join PC,
then $PC^2 = OP^2 + OC^2 = 1 + 3 = 4$

$$PC = \sqrt{4}$$

The lengths of PA, PB, PC may now be found *by measurement*, and by continuing the process we may find $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$,

EXERCISES ON THEOREMS 29, 30 (*Continued*)

13 Prove the following formula

Diagonal of square = side $\times \sqrt{2}$

Hence find to the nearest centimetre the diagonal of a square on a side of 50 metres

Draw a plan (scale 1 cm to 10 metres) and obtain the result as nearly as you can by measurement

14 ABC is an equilateral triangle of which each side $= 2m$ units, and the perpendicular from any vertex to the opposite side $= p$

Prove that $p = m\sqrt{3}$

Test this result graphically, when each side $= 8$ cm.

15 If in a triangle $a=m^2-n^2$, $b=2mn$, $c=m^2+n^2$, prove algebraically that $c^2=a^2+b^2$

Hence by giving various numerical values to m and n, find sets of numbers representing the sides of right-angled triangles

16 In a triangle ABC, AD is drawn perpendicular to BC Let p denote the length of AD

(i) If $a=25$ cm, $p=12$ cm, BD$=9$ cm, find b and c.

(ii) If $b=41''$, $c=50''$, BD$=30''$, find p and a

And prove that $\sqrt{b^2-p^2}+\sqrt{c^2-p^2}=a$

17 In the triangle ABC, AD is drawn perpendicular to BC Prove that

$$c^2-\text{BD}^2=b^2-\text{CD}^2$$

If $a=51$ cm, $b=20$ cm, $c=37$ cm, find BD

Thence find p, the length of AD, and the area of the triangle ABC

18 Find by the method of the last example the areas of the triangles whose sides are as follows

(i) $a=17''$, $b=10''$, $c=9''$ (ii) $a=25$ ft, $b=17$ ft, $c=12$ ft

(iii) $a=41$ cm, $b=28$ cm, $c=15$ cm (iv) $a=40$ yd, $b=37$ yd, $c=13$ yd.

19 A straight rod PQ slides between two straight rulers OX, OY placed at right angles to one another In one position of the rod OP$=5\ 6$ cm, and OQ$=3\ 3$ cm If in another position OP$=4{\cdot}0$ cm, find OQ graphically, and test the accuracy of your drawing by calculation.

20 ABC is a triangle right-angled at C, and p is the length of the perpendicular from C on AB By expressing the area of the triangle in two ways, shew that

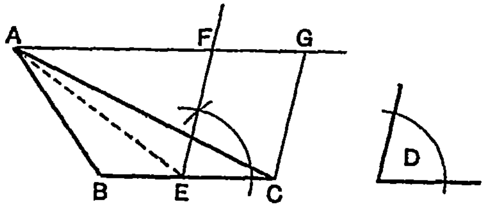
$$pc=ab$$

Hence deduce
$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$$

PROBLEMS ON AREAS

√ PROBLEM 17

To describe a parallelogram equal to a given triangle, and having one of its angles equal to a given angle



Let ABC be the given triangle, and D the given angle

It is required to describe a parallelogram equal to ABC, *and having one of its angles equal to* D

Construction. Bisect BC at E

At E in CE, make the ∠CEF equal to D,

through A draw AFG par' to BC,

and through C draw CG par' to EF

Then FECG is the required par^{m}

Proof Join AE

Now the △^{s} ABE, AEC are on equal bases BE, EC, and of the same altitude,

the △ABE = the △AEC

the △ABC is double of the △AEC

But FECG is a par^{m} by construction,

and it is double of the △AEC,

being on the same base EC, and between the same par^{ls} EC and AG

the par^{m} FECG = the △ABC,

and one of its angles, namely CEF, = the given ∠D

EXERCISES

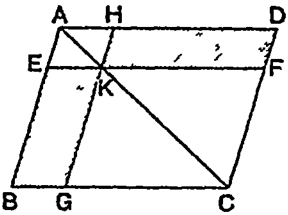
(Graphical.)

1. Draw a square on a side of 5 cm., and make a parallelogram of equal area on the same base, and having an angle of 45°.

Find (i) by calculation, (ii) by measurement the length of an oblique side of the parallelogram.

2. Draw any parallelogram ABCD in which AB=2½" and AD=2", and on the base AB draw a rhombus of equal area.

DEFINITION. In a parallelogram ABCD, if through any point K in the diagonal AC parallels EF, HG are drawn to the sides, then the figures EH, GF are called **parallelograms about AC**, and the figures EG, HF are said to be their **complements**.



3. *In the diagram of the preceding definition shew by Theorem* 21 *that the complements* EG, HF *are equal in area.*

Hence, given a parallelogram EG, and a straight line HK, deduce a construction for drawing on HK as one side a parallelogram equal and equiangular to the parallelogram EG.

4. Construct a rectangle equal in area to a given rectangle CDEF, and having one side equal to a given line AB.

If AB=6 cm., CD=8 cm., CF=3 cm., find by measurement the remaining side of the constructed rectangle.

5. Given a parallelogram ABCD, in which AB=2.4", AD=1.8", and the ∠A=55°. Construct an equiangular parallelogram of equal area, the greater side measuring 2.7". Measure the shorter side.

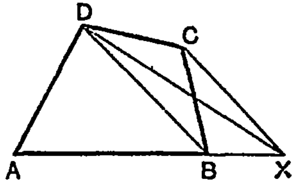
Repeat the process giving to A any other value, and compare your results. What conclusion do you draw?

6. Draw a rectangle on a side of 5 cm. equal in area to an equilateral triangle on a side of 6 cm.

Measure the remaining side of the rectangle, and calculate its approximate area.

PROBLEM 18

To draw a triangle equal in area to a given quadrilateral.



Let ABCD be the given quadrilateral

It is required to describe a triangle equal to ABCD *in area*

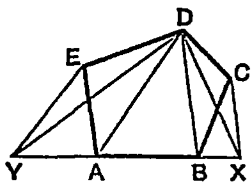
Construction. Join DB
Through C draw CX par[l] to DB, meeting AB produced in X.
Join DX
Then DAX is the required triangle

Proof Now the $\triangle^s$ XDB, CDB are on the same base DB and between the same par[ls] DB, CX,

the $\triangle$ XDB = the $\triangle$ CDB in area

To each of these equals add the $\triangle$ ADB,
then the $\triangle$ DAX = the fig ABCD

COROLLARY In the same way it is always possible to draw a rectilineal figure equal to a given rectilineal figure, and having fewer sides by one than the given figure, and thus step by step, any rectilineal figure may be reduced to a triangle of equal area.

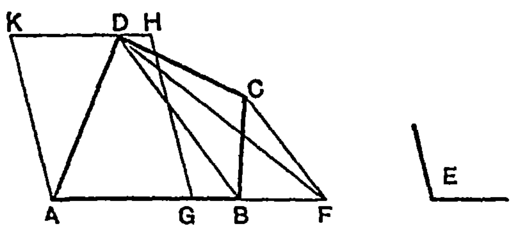


For example, in the adjoining diagram the five-sided fig EDCBA is equal in area to the four-sided fig EDXA

The fig EDXA may now be reduced to an equal $\triangle$ DXY

√ PROBLEM 19

To draw a parallelogram equal in area to a given rectilineal figure, and having an angle equal to a given angle



Let ABCD be the given rectil fig, and E the given angle

It is required to draw a par^m equal to ABCD *and having an angle equal to* E

Construction Join DB

Through C draw CF par^l to DB, and meeting AB produced in F

Join DF

Then the △ DAF = the fig ABCD *Prob* 18

Draw the par^m AGHK equal to the △ ADF, and having the ∠ KAG equal to the ∠ E *Prob* 17

Then the par^m KG = the △ ADF

= the fig ABCD,

and it has the ∠ KAG equal to the ∠ E

NOTE If the given rectilineal figure has more than four sides, it must first be reduced, step by step, until it is replaced by an equivalent triangle.

EXERCISES

(Reduction of a Rectilineal Figure to an equivalent Triangle)

Draw a quadrilateral ABCD from the following data

AB=BC=5 5 cm , CD=DA=4 5 cm , the $\angle$A=75°

Reduce the quadrilateral to a triangle of equal area Measure the base and altitude of the triangle, and hence calculate the approximate area of the given figure.

2 Draw a quadrilateral ABCD having given

AB=2 8″, BC=3·2″, CD=3 3″, DA=3 6″, and the diagonal BD=3 0″

Construct an equivalent triangle, and hence find the approximate area of the quadrilateral

3 On a base AB, 4 cm in length, describe an equilateral pentagon (5 sides), having each of the angles at A and B 108°

Reduce the figure to a triangle of equal area, and by measuring its base and altitude, calculate the approximate area of the pentagon.

4 *A quadrilateral field ABCD has the following measurements*

AB=450 metres, BC=380 m , CD=330 m , AD=390 m.,

and the diagonal AC=660 m

Draw a plan (scale 1 cm to 50 metres) Reduce your plan to an equivalent triangle, and measure its base and altitude Hence estimate the area of the field

(Problems State your construction, and give a theoretical proof)

5 Reduce a triangle ABC to a triangle of equal area having its base BD of given length (D lies in BC, or BC produced.)

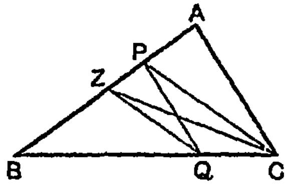
6 Construct a triangle equal in area to a given triangle, and having a given altitude

7 ABC is a given triangle, and X a given point Draw a triangle equal in area to ABC, having its vertex at X, and its base in the same straight line as BC

8 Construct a triangle equal in area to the quadrilateral ABCD, having its vertex at a given point X in DC, and its base in the same straight line as AB

9 Shew how a triangle may be divided into n equal parts by straight lines drawn through one of its angular points

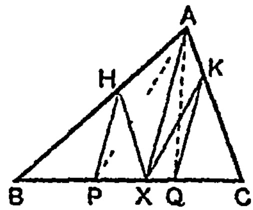
10 *Bisect a triangle by a straight line drawn through a given point in one of its sides*



[Let ABC be the given $\triangle$, and P the given point in the side AB

Bisect AB at Z, and join CZ, CP
Through Z draw ZQ parallel to CP
Join PQ.
Then PQ bisects the $\triangle$]

11 *Trisect a triangle by straight lines drawn from a given point in one of its sides*



[Let ABC be the given $\triangle$, and X the given point in the side BC

Trisect BC at the points P, Q. *Prob* 7
Join AX, and through P and Q draw PH and QK parallel to AX
Join XH, XK
These straight lines trisect the $\triangle$, as may be shewn by joining AP, AQ]

12 Cut off from a given triangle a fourth, fifth, sixth, or any part required by a straight line drawn from a given point in one of its sides

13 *Bisect a quadrilateral by a straight line drawn through an angular point*

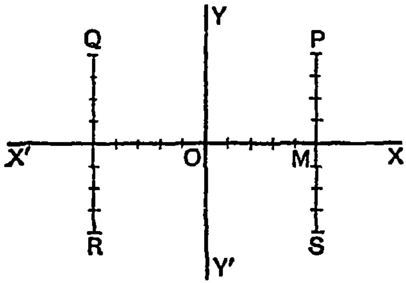
[Reduce the quadrilateral to a triangle of equal area, and join the vertex to the middle point of the base]

14 Cut off from a given quadrilateral a third, a fourth a fifth, or any part required, by a straight line drawn through a given angular point

AXES OF REFERENCE COORDINATES

EXERCISES FOR SQUARED PAPER

If we take two fixed straight lines XOX′, YOY′ cutting one another at right angles at O, the position of any point P with reference to these lines is known when we know its distances from each of them Such lines are called axes of rèference, XOX′ being known as the axis of x, and YOY′ as the axis of y Their point of intersection O is called the origin.



The lines XOX′, YOY′ are usually drawn horizontally and vertically

In practice the distances of P from the axes are estimated thus

From P, PM is drawn perpendicular to X′X, and OM and PM are measured

OM is called the abscissa of the point P, and is denoted by x
PM " ordinate " " " y
The abscissa and ordinate taken together are called the coordinates of the point P, and are denoted by (x, y)

We may thus find the position of a point if its coordinates are known

EXAMPLE *Plot the point whose coordinates are* (5, 4)
Along OX mark off OM, 5 units in length
At M draw MP perp to OX, making MP 4 units in length.
Then P is the point whose coordinates are (5, 4)

The axes of reference divide the plane into four regions XOY, YOX′, X′OY′, Y′OX, known respectively as the first, second, third, and fourth quadrants

It is clear that in each quadrant there is a point whose distances from the axes are equal to those of P in the above diagram, namely, 5 units and 4 units

The coordinates of these points are distinguished by the use of the *positive* and *negative* signs, according to the following system

Abscissæ measured along the x-axis to the right of the origin are positive, those measured to the left of the origin are negative Ordinates which lie above the x-axis (that is, in the first and second quadrants) are positive, those which lie below the x-axis (that is, in the third and fourth quadrants) are negative

Thus the coordinates of the points Q, R, S are

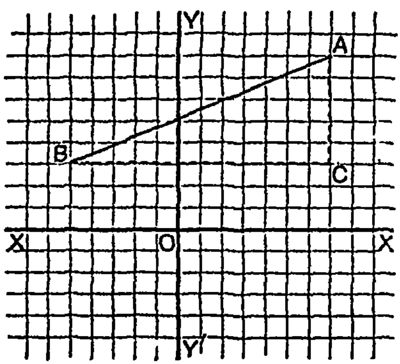
$$(-5, 4),\ (-5, -4),\ \text{and}\ (5, -4)\ \text{respectively}$$

NOTE. The coordinates of the origin are (0, 0)

In practice it is convenient to use squared paper Two intersecting lines should be chosen as axes, and slightly thickened to aid the eye, then one or more of the length divisions may be taken as the linear unit The paper used in the following examples is ruled to tenths of an inch

EXAMPLE 1 *The coordinates of the points* A *and* B *are* (7, 8) *and* (-5, 3) *plot the points and find the distance between them.*

After plotting the points as in the diagram, we may find AB approximately by direct measurement.



Or we may proceed thus

Draw through B a line par[l] to XX′ to meet the ordinate of A at C Then ACB is a rt angled Δ in which BC=12, and AC=5

Now $AB^2 = BC^2 + AC^2$

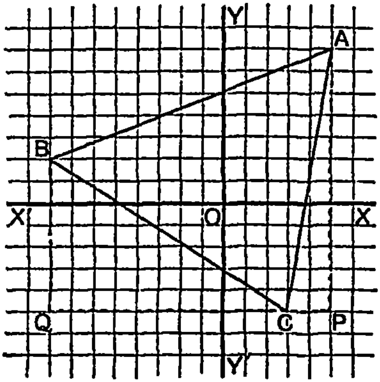
$= 12^2 + 5^2$

$= 144 + 25$

$= 169$

$\therefore$ AB=13

EXAMPLE 2 *The coordinates of* A, B, *and* C *are* (5, 7), (−8, 2), *and* (3, −5), *plot these points and find the area of the triangle of which these are the vertices*



Having plotted the points as in the diagram, we may measure AB, and draw and measure the perp from C on AB Hence the approximate area may be calculated.

Or we may proceed thus

Through A and B draw AP, BQ par[l] to YY′

Through C draw PQ par[l] to XX′

Then the $\triangle$ ABC = the trap[m] APQB − the two rt angled $\triangle$[s] APC, BQC

$$= \tfrac{1}{2}PQ(AP+BQ) - \tfrac{1}{2}\ AP\ PC - \tfrac{1}{2}\ BQ\ QC$$
$$= \tfrac{1}{2}\times 13\times 19 \quad - \tfrac{1}{2}\times 12\times 2 \quad - \tfrac{1}{2}\times 7\times 11$$
$$= 73 \text{ units of area}$$

EXERCISES

1. Plot the following sets of points

(i) (6, 4), (−6, 4), (−6, −4), (6, −4),
(ii) (8, 0), (0, 8), (−8, 0), (0, −8),
(iii) (12, 5), (5, 12), (−12, 5), (−5, 12)

2 Plot the following points, and shew experimentally that each set lie in one straight line

(i) (9, 7), (0, 0), (−9, −7), (ii) (−9, 7), (0, 0), (9, −7)

Explain these results theoretically

3 Plot the following pairs of points, join the points in each case, and measure the coordinates of the mid-point of the joining line.

(i) (4, 3), (12, 7), (ii) (5, 4), (15, 16)

Shew *why* in each case the coordinates of the mid point are respectively *half the sum of the abscissæ* and *half the sum of the ordinates* of the given points

4 Plot the following pairs of points, and find the coordinates of the mid point of their joining lines

(i) (0, 0), (8, 10), (ii) (8, 0), (0, 10),
(iii) (0, 0), (−8, −10), (iv) (−8, 0), (0, −10)

5 Find the coordinates of the points of trisection of the line joining (0, 0) to (18, 15)

6 Plot the two following series of points

(i) (5, 0), (5, 2), (5, 5), (5, −1), (5, −4),
(ii) (−4, 8), (−1, 8), (0, 8), (3, 8), (6, 8)

Shew that they lie on two lines respectively parallel to the axis of y, and the axis of x Find the coordinates of the point in which they intersect

7 Plot the following points, and calculate their distances from the origin

(i) (15, 8), (ii) (−15, −8), (iii) (2 4″, 7″), (iv) (− 7″, 2 4″)
Check your results by measurement

8 Plot the following pairs of points, and in each case calculate the distance between them

(i) (4, 0), (0, 3), (ii) (9, 8), (5, 5),
(iii) (15, 0), (0, 8), (iv) (10, 4), (−5, 12),
(v) (20, 12), (−15, 0), (vi) (20, 9), (−15, −3)
Verify your calculation by measurement

9 Shew that the points (−3, 2), (3, 10), (7, 2) are the angular points of an isosceles triangle. Calculate and measure the lengths of the equal sides

10 Plot the eight points (0, 5), (3, 4), (5, 0), (4, −3), (−5, 0), (0, −5), (−4, 3), (−4, −3), and shew that they all lie on a circle whose centre is the origin

11 Explain by a diagram why the distances between the following pairs of points are all equal

(i) $(a, 0)$, $(0, b)$, (ii) $(b, 0)$, $(0, a)$, (iii) $(0, 0)$, (a, b)

12. Draw the straight lines joining

(i) $(a, 0)$ and $(0, a)$, (ii) $(0, 0)$ and (a, a),

and prove that these lines bisect each other at right angles

13 Shew that (0, 4), (12, 9), (12, −4) are the vertices of an isosceles triangle whose base is bisected by the axis of x

14 Three vertices of a rectangle are (14, 0), (14, 10), and (0, 10) find the coordinates of the fourth vertex, and of the intersection of the diagonals

15 Prove that the four points (0, 0), (13, 0) (18, 12), (5, 12) are the angular points of a rhombus Find the length of each side, and the coordinates of the intersection of the diagonals

16 Plot the locus of a point which moves so that its distances from the points (0, 0) and (4, −4) are always equal to one another Where does the locus cut the axes?

17 Shew that the following groups of points are the vertices of rectangles Draw the figures, and calculate their areas

(i) (4, 3), (17, 3), (17, 12), (4, 12),
(ii) (3, 2), (3, 15), (−6, 15), (−6, 2),
(iii) (5, 1), (−8, 1), (−8, −8), (5, −8).

18 Join in order the points (1″, 0), (0 1″), (−1″, 0), (0, −1″) Of what kind is the quadrilateral so formed? Find its area

If a second figure is formed by joining the middle points of the first, find its area

19 Plot the triangles given by the following sets of points, and find their areas.

(i) (10, 10), (4, 0), (18, 0), (ii) (10, −10), (4, 0), (18, 0),
(iii) (−10, 10), (−4, 0), (−18, 0), (iv) (−10, −10), (−4, 0), (−18, 0)

20 Draw the triangles given by the points

(i) (0, 0), (5, 3), (6, 0), (ii) (0, 0), (3, 0), (0, 6)

Find their areas, and measure the angles of the first triangle

21 Plot the triangles given by the following sets of points Shew that in each case one side is parallel to one of the axes Hence find the area

(i) (0, 0), (12, 10), (12, −6), (ii) (0, 0), (5, 8), (−15, 8),
(iii) (0, 0), (−12, 12), (−12, −8), (iv) (0, 0), (−6, −8), (20, −8)

22 In the following triangles shew that two sides of each are parallel to the axes Find their areas

(i) (5, 5), (15, 5), (15, 15), (ii) (8, 3), (8, 18), (0, 18),
(iii) (4, 8), (−16, −4), (4, −4), (iv) (1, 15), (−11, 15), (1, −7)

23 Shew that (−5, 5), (7, 10), (10, 6), (−2, 1) are the angular points of a parallelogram. Find its sides and area

24 Shew that each of the following sets of points gives a trapezium Find the area of each

(i) (3, 0), (3, 3), (9, 0), (9, 6), (ii) (0, 3), (−5, 3), (−2, −3), (0, −3),
(iii) (8, 4), (4, 4), (11, −1), (3, −1), (iv) (0, 0), (−1, 5), (−4, 5), (−8, 0)

25 Find the area of the triangles given by the following points

(i) (5, 5), (20, 10), (12, 14), (ii) (7, 6), (−10, 4), (−4, −3);
(iii) (0, −6), (0, −3), (14, 5), (iv) (6, 4), (−7, −6), (−2, −15)

26 Shew that (−5, 0), (7, 5), (19, 0), (7, −5) are the angular points of a rhombus Find its sides and its area

27 Join the points (0, −5), (12, 0), (4, 6), (−8, −3), in the order given Calculate the lengths of the first three sides and measure the fourth Find the areas of the portions of the figure lying in the first and fourth quadrants.

28 The coordinates of four points A, B, C, D are respectively

(−4, −4), (−10, 4), (−10, 13) (5, 5)

Calculate the lengths of AB, BC, CD, and measure AD Also calculate the area of ABCD by considering it as the difference of two triangles

29 Draw the figure whose angular points are given by

(0, −3), (8, 3), (−4, 8), (−4, 3), (0, 0)

Find the lengths of its sides, taking the points in the above order Also divide it into three right-angled triangles, and hence find its area

30 A plan of a triangular field ABC is drawn on squared paper (scale 1″=100 yds) On the plan the coordinates of A, B, C are (−1″, −3″), (3″, 4″), (−5″, −2″) respectively Find the area of the field, the length of the side represented by BC, and the distance from this side of the opposite corner of the field

31 Shew that the points (6, 0), (20, 6), (14, 20), (0, 14) are the vertices of a square Measure a side and hence find the approximate area. Calculate the area exactly (i) by drawing a circumscribing square through its vertices, (ii) by subdividing the given square as in the first figure on page 120

MISCELLANEOUS EXERCISES

1 AB and AC are unequal sides of a triangle ABC, AX is the median through A, AP bisects the angle BAC, and AD is the perpendicular from A to BC Prove that AP is intermediate in position and magnitude to AX and AD

2 In a triangle if a perpendicular is drawn from one extremity of the base to the bisector of the vertical angle, (i) it will make with either of the sides containing the vertical angle *an angle equal to half* the sum of the angles at the base, (ii) it will make with the base an angle equal to half the difference of the angles at the base

3 In any triangle the angle contained by the bisector of the vertical angle and the perpendicular from the vertex to the base is equal to half the difference of the angles at the base.

4 Construct a right-angled triangle having given the hypotenuse and the difference of the other sides

5 Construct a triangle, having given the base, the difference of the angles at the base, and (i) the difference, (ii) the sum of the remaining sides

6 Construct an isosceles triangle, having given the base and the sum of one of the equal sides and the perpendicular from the vertex to the base

7 Shew how to divide a given straight line so that the square on one part may be double of the square on the other

8 ABCD is a parallelogram, and O is any point without the angle BAD or its opposite vertical angle, shew that the triangle OAC is equal to the sum of the triangles OAD, OAB

If O is within the angle BAD or its opposite vertical angle, shew that the triangle OAC is equal to the difference of the triangles OAD, OAB

9 The area of a quadrilateral is equal to the area of a triangle having two of its sides equal to the diagonals of the given figure, and the included angle equal to either of the angles between the diagonals.

10 Find the locus of the intersection of the medians of triangles described on a given base and of given area.

11 On the base of a given triangle construct a second triangle equal in area to the first, and having its vertex in a given straight line

12 ABCD is a parallelogram made of rods connected by hinges If AB is fixed, find the locus of the middle point of CD

ANSWERS TO NUMERICAL EXERCISES

Since the utmost care cannot ensure absolute accuracy in graphical work, results so obtained are likely to be only approximate. The answers here given are those found by calculation, and being true so far as they go, furnish a standard by which the student may test the correctness of his drawing and measurement. Results within one per cent. of those given in the Answers may usually be considered satisfactory.

Exercises Page 15

1 30°, 126°, 261°, 85°. 11 min., 37 min.

2 $112\frac{1}{2}$°, 155°, 5 hrs. 45 min. 3 50°, 8 hrs. 40 min.

4. (i) 145°, 35°, 145° (ii) 55°, 55° 86°, 94°

Exercises Page 27

1 68°, 37°, 75°, nearly 2 6·0 cm 4. 22°, 50°, 73° nearly

5 37 ft 6 101 metres 7 27 ft 8 424 yds., nearly, N.W.

9 281 yds., 155 yds., 153 yds. 10 214 yds.

Exercises Page 41

1 125°, 55°, 125° 12 15 secs., 30 secs.

Exercises Page 43

3 21° 4. 27° 5 92°, 46° 6 67°, 62°

Exercises Page 45

1 30°, 60°, 90° 2 (i) 36°, 72°, 72°, (ii) 20°, 80°, 80°

3 40° 4 51°, 111°, 18 5 (i) 34°, (ii) 107°

6 68 7 120° 8 36°, 72°, 108°, 144°

9 165° 11 5, 15

Exercises Page 47

2 (i) 45°, (ii) 36° 3 (i) 12, (ii) 15

Exercises Page 54

4. (i) 81°, c. (ii) 55°

10

Degrees	15	30	45°	60°	75°
Cm.	4·1	4·6	5·7	8·0	15·6

11

Degrees	0°	30°	60°	90°	120°	150°	180°
Cm.	1·0	2·0	3·6	5·0	6·1	6·8	7·0

12 37 ft. 13 112 ft. 14. 346 yds. 693 yds.

Exercises Page 61

14. 54°, 72°, 54° 15 36° 16 4

18 (i) 16, (ii) 45°, (iii) $11\frac{1}{4}$° per sec

Exercises Page 68

2 6 80 cm 3 2·24″ 4 0 39 5 2 54 8 10 6 cm.

9 3 35″ 10 20 miles, 12 6 km

11 147 miles, 235 km 1 cm represents 22 km

12 1″ represents 15 mi, 1″ represents 20 mi

Exercises Page 79

3 0 53 in 4. 1 3 cm. 5 2 4″

Exercises Page 84.

1 4 3 cm, 5 2 cm, 6 1 cm 2 1 10 3 200 yards

4. 65°, 77 m, 61 m, 56 m 5 6 04 knots S, 15° E, nearly

6 Results equal. 9 cm 7 4 3 cm, 9 8 cm, 60°, 120°

8 (i) One solution, (ii) two, (iii) one, right-angled, (iv) impossible

9 380 yds 10 6 5 cm 11 6 9 cm

12 Two solutions, 10 4 cm or 4 5 cm 16 2 8 cm, 4 5 cm, 5 3 cm

18 5 8 cm, 4 2 cm. 19 7 cm, 8 cm

Exercises Page 89

1 60°, 120° 2 3 54″ 3 2 12″ 4 4 4 cm

5 16 4 cm, 3 4% 6 90° 7 (i) 4·25″, (ii) $B=D=90°$

Exercises Page 102

1 6 sq in 2 6 sq in 3 2 80 sq in 4. 3 50 sq in

5 3 30 sq in 6 3 36 sq in 7 198 sq m 8 42 sq ft

9 10,000 sq m 10 110 sq ft 11 5 cm. 12 2 6 in

14. 900 sq yds, 48 yds, 4 8″ 15 11700 sq m

16 1 cm = 10 yds 17 1 6″ 18 600 sq ft 19 1152 sq ft.

20 100 sq ft 21 156 sq ft 22 110 sq ft

23 288 sq ft 24. 72 sq ft 25 75 sq ft

Exercises Page 105

1 (i) 22 cm, (ii) 3 6″ 2 3 4 sq in 3 574 5 sq in

4. 1 5″ 5 1·93″, 75°

Exercises Page 107

1 (i) 180 sq ft, (ii) 8 4 sq in, 1 hectare
2 (i) 13 44 sq cm, (ii) 15 40 sq cm, (iii) 20 50 sq cm
3 15 sq cm 4. 6 3 sq in
5 (i) 8″, (ii) 13 cm 6 3 36 sq in

Exercises Page 110

1 11400 sq yds 2 6312 sq m
3 2 4 cm, 5 1 cm 4 2 04″, 2 20″

5

Angle	0°	30°	60°	90°	120°	150°	180°
Area in sq cm	0	7 5	13 0	15 0	13 0	7 5	0

Exercises Page 111

1 66 sq ft 2 84 sq yds 3 126 sq m
4. 132 sq cm 5 180 sq ft 6 306 sq m

Exercises Page 113

1 6 sq in 2 170 sq ft 3 615 sq m 4 8 4 sq in
5 31·2 sq cm 6 5·20 sq in 7 24 sq cm

Exercises Page 115

1 (i) 25 5 sq cm, (ii) 15 6 sq cm
2 (i) 8·95 sq in, (ii) 9 5 sq in 3 12500 sq m

Exercises Page 116

4. 3 3 sq in 5 7 5 cm 6 3 6 sq in

Exercises Page 121

1 (i) 5 cm, (ii) 6 5 cm, (iii) 3 7″ 2 (i) 1 6″, (ii) 2 8 cm
3 41 ft 4. 65 miles 5 6 1 km 6 16 ft
7 48 m 8 25 miles 9 73 m 10 62 ft

Exercises Page 123

10 (i) and (iii) 11 2 83″ 12 4·24 cm, 18 sq cm
13 70 71 sq m 14 $p=6$ 93 cm
16 (i) 20 cm, 15 cm (ii) 40 cm, 39 cm
17 35 cm, 12 cm, 306 sq cm
18 (i) 36 sq in, (ii) 90 sq ft, (iii) 126 sq cm, (iv) 240 sq yds
19 5 1 cm nearly

Exercises Page 127

1 7 1 cm 4 4 0 cm 5 1 6″ 6 3 1 cm , 15 6 sq cm

Exercises Page 130

1 23·90 sq cm 2 8 40 sq in
3 27 52 sq cm 4 129800 sq m

Exercises Page 134.

3 (i) (8, 5) , (ii) (10, 10)
4 (i) (4, 5) , (ii) (4, 5) , (iii) (−4, −5) , (iv) (−4, −5)
5 (6, 5), (12, 10) 6 (5, 8)
7 (i) 17 , (ii) 17 , (iii) 2 5″ , 2 5″
8 (i) and (ii) 5 , (iii) and (iv) 17 , (v) and (vi) 37 9 10
14. (0, 0) (7, 5) 15 13 , (9, 6)
16 A straight line passing through the points (4, 0), (0, −4)
17 117 units of area in each case 18 A square 2 sq in 1 sq in
19 Each = 70 units of area 20 9 units of area. 31°, 71°, 78°
21 (i) 96 , (ii) 80 , (iii) 120 , (iv) 104
22 (i) 50 , (ii) 60 , (iii) 120 , (iv) 132
23 Sides 5, 13 ; area 63 24. (i) 27 , (ii) 21 , (iii) 30 , (iv) 27 5
25 (i) 50 , (ii) 65 5 , (iii) 21 , (iv) 83 5
26 Each side 13 , area 120 27 13, 10, 15, 8 24, 42, 30
28 AB = 10, BC = 9, CD = 17, DA = 12 7 Area = 130 5
29 10, 13, 5, 5, 3 Area = 60 30 160,000 sq yds 1000 yds 320 yds
31 Side = 15 23 , area = 232 units of area

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